KNOT THEORY

FINAL PROJECT REPORT

OF

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BY

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PREFACE

A brief survey of the vast field of Knot Theory is given in this final report named **Knot Theory.** The subject matter is divided into six chapters.

The **First Chapter** introduces the path which led to the study of the mathematical theory of knots and a brief history of the subject with the relevance of research in it.

In the **Second Chapter** some basic definitions of mathematical knots, their equivalence and a few classical knot invariants used to distinguish knots are dealt with.

The **Third Chapter** relates to the tabulation of knots. In this chapter, methods of representing different knots, their classification as also brief discussion on tangles are made.

Chapter Four deals with a special mention on Seifert surfaces and finding the genus of knots. A detail of compact oriented surface, surfaces with boundary, the Euler characteristic and genus of a surface are also discussed.

Polynomial invariants of knots are dealt with in the **Fifth Chapter**.

Sixth Chapter relates to applications of knot theory with special emphasis on application to DNA replication, molecular synthesis and chirality of molecules.

I do not claim any originality in this report, but the presentation of the subject matter is mine and I have made every attempt to make it unique. Reference has been made from many books as also many published papers (list is given in the reference) while preparing this project report. Discussions with scientists working in this field have also contributed in this endeavour.

Place: Ottapalam

Date: SUNITHA K G

ACKNOWLEDGEMENT

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Sunitha K G

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Chapter- 1

INTRODUCTION

Man has been fascinated by Knots since time immemorial. He has been using different types of knots for different purposes. He used knots to make nets, tie things together, weave clothes and mat; construct bridges, climb rocks and so on. Beautiful decorative items and embroidery works are being made using knots. The Inca dynasty of South America used *Quipu* (word meaning knot) to store information to be sent to different places. Man's interest in knots gave it a symbolic significance in certain cultures and spirituality. A link called the *Borromean Rings* was a symbol of the *Borromeo family*, an aristocratic family of Northern Italy. The *endless knot* or *eternal knot* is a symbolic knot important as a cultural marker in Tibet. In India, a ring made of *Dharbha Grass* is worn at the time of certain rituals. The ring is made by a special knot called the *Pavithrakettu*. A scout boy plays with a rope tying many kinds of knots like the *overhand knot, the Reef knot, the figure eight knot, the clove hitch, the fisherman knot, the square knot* and else. When Alexander the Great cut the highly complicated *Gordian knot* with his sword to untie the knot, it found its place in English literature as a metaphor for an intractable problem solved easily by loopholes.

Though the history of knots predates invention of writing, the arrival of knots to the field of mathematics is a later development. What fascinated mathematicians so much to take up the study of knots? In fact, the motivation was initiated by the application of its theory in chemistry, later on in physics and more recently in biology as well.

§. 1.1. Historical Background

The first reference to knots from a mathematical perspective comes perhaps in 1771 in a paper titled *"Remarques sur les problems de situation"* written by a French Mathematician *Theophil Vandermonde*.

The study of mathematical theory of knots as now referred to as Knot Theory can be traced back to the 19th century when the German Mathematician, *Carl Friedrich Gauss* created a method for tabulation of knots. *Gauss* applied the mathematical concept of knots for his work in electro dynamics. He wanted to know how much work was done on a magnetic pole along a closed curve in the presence of a loop of current. He considered two non intersecting loops. He was able to get an answer. In the process, he discovered what is now referred to as *"Gauss linking number"*. This number is invariant under *ambient isotopy*. It became the first method to distinguish two non equivalent links from each other. Inspired by the work of *Gauss*, his student, *Johann Benedict Listing* became interested in knots. In his work

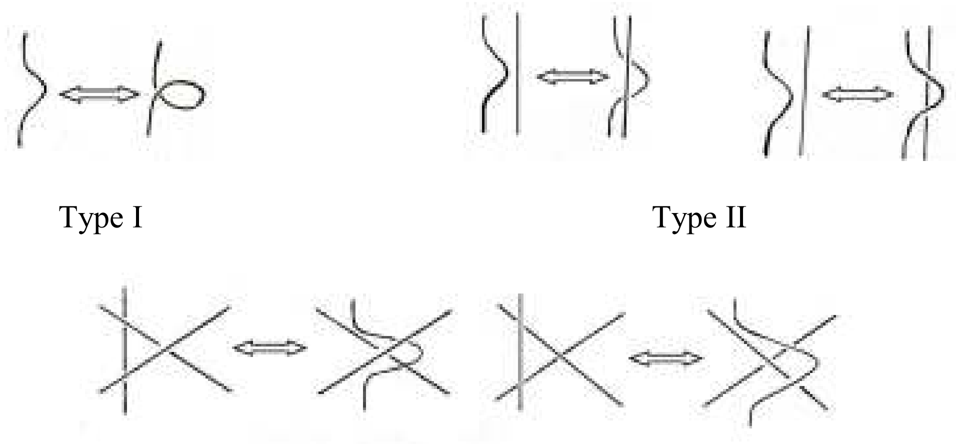
"Vorstudien zur Topologie" in which he first coined the term *"Topology"*, he included a discussion on mathematical knots and their classification. His interest was in chirality of knot, that is, the equivalence of a knot to its mirror image. The significant result in his paper was the statement that the left and right *Trefoil Knots* are different, that is, *Trefoil Knot* is not *amphichiral*. He also stated that figure- eight knot is *amphichiral*. In 1860s the famous English Physicist, *Sir William Thomson (*later on known as *Lord Kelvin)* was inspired by the physicist, *Herman von Helmholtz*'s work on Vortex Motion and demonstration of producing vortex smoke rings by *Peter Gurthrie Tait*. Thomson presented a paper proposing that atoms were knotted vortices. Vortex atoms were 3- dimensional knotted tubes of *ether*. According to the vortex atom theory, molecules could be thought of as intertwined vortex atoms which would resemble a link. The physicist, *James Clerk Maxwell*, a friend of *Thomson* became interested in knots and used it in the study of electricity and magnetism. He created knot diagrams specifying over and under crossings. He defined the 3 *Reidemeister moves*.

The vortex model of atomic theory necessitated the classification of knots. The physicist *Peter Gurthrie Tait* began making the first table of knots in 1867. Later on *Thomson Kirkman* made the first major contribution to the task of classifying knots. He made a table of diagrams for alternating knots with up to 11 crossings. During the process he realised that he had to reduce the knot diagram to a minimum number of crossings to avoid duplication. *Tait* and *Charles Newton Little* corrected *Kirkman's Table* and published the first official table of distinct alternating knots up to 10 crossings. Later on when the vortex model of atoms was proven wrong, the Knot Theory was forgotten.

In 1900 an important development took place in the research of knots when *Julus Henry Poincare* introduced the fundamental group which laid the foundation for Algebraic Topology. In 1908 *Henreich Tietze* used the fundamental group of the exterior of a knot in R^3 called the ***knot group***, to distinguish the unknot from the trefoil knot. *Wilhelm Wirtinger* gave a method of finding a knot group presentation in 1905. *Max Dehn* refined the notion of knot group, developed an algorithm to construct the fundamental group of the complement of a link and showed that the trefoil knot is not amphichiral. Thus slowly the research in knots was taken up by mathematicians.

In the 1920s, mathematicians became more interested in knot theory. This became a research area and began to be applied to purely mathematical concepts. *Braid Theory* was developed by *Emil Artin* in early 1920s. The application of this theory ranged from quantum mechanics to combinatorics to the textile industry. Knot Theory regained its importance through the braid theory.

Tait was the first to construct a relation between knots and planar graphs. In the 1920s *Kurt Reidemeister* worked on planar diagrams of knots. He proved that "Two knots K *and* K' with diagrams D, D' are equivalent if and only if their diagrams are related by definite sequences $D = D_0, D_1, \dots, D_n = D'$ of intermediate diagrams such that each differs from its predecessor by one of the following 3 *Reidemeister moves."*



Type I Type II

Type III

Figure 1.1 (Reidemeister moves)

Reidemeister used the fundamental groups to prove this. During the same time James Waddell Alexander proved this using concept of the homology groups. Later, J.W. Alexander proved that every link can be represented as a closed braid. He discovered a polynomial invariant in 1928. This helped to distinguish many non isotopic knots from one another.

The first book written on *Knot Theory* entitled *Knottentheorie* was by *Reidemeister*.

In the 1940s a mathematician, *Horst Schubert* invented the invariant called the *Bridge Number* of a knot and proved some important results of *Bridge Number*.

Ralph H Fox, a mathematician from USA, proposed to replace polygonal curves in knots by an appropriate topologically defined set of curves and R^3 by compact three – manifold. This reshaped the foundations of knot theory and provided a greater access to the tools of topology for studying knots. His work led to a number of new geometric invariants. Knot group using the fundamental group concept was focussed upon.

In the 1960s, *John Conway* developed a method for knot notation and used it to find the *Alexander Polynomial*. He defined *Tangles* to explain the notation. *Dowker* and *Thistle Waite* were the first to computerise *Knot Enumeration*. This helped to expand the *Knot Table* of 11 crossing to 13 crossings.

In 1984, *Vaughan Jones* discovered an invariant called the Jones polynomial. This was the first polynomial invariant that enabled to distinguish the *Right hand Trefoil Knot* from the *Left hand Trefoil Knot*. Four months later after the discovery of the *Jones Polynomial*, an invariant called the *HOMFLY polynomial* was discovered by six mathematicians – *Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter*.

In 1985 a mathematician *Louis Kauffman* discovered another approach to the *Jones Polynomial* using what is called a *Bracket Polynomial*.

Towards the end of the 1980s, V A Vassiliev introduced an invariant which could be used to study the Jones invariant more systematically and could be given a topological interpretation.

In Japan, from around 1960s, many contributions were made to knot theory, initiated by the mathematician H. Terasaka. T. Homma, S. Kinoshita (later, moving to U.S.A.) and K. Murasugi (later, moving to Canada), F. Hosokawa, were some others. An international conference on "Knot theory and related topics" was held at Osaka as a satellite conference of ICM Kyoto in 1990. The proceeding of this conference was "Knots 90" by Akio Kawauchi, e.d. by Walter de Gruyter.

In 1992, the Journal of Knot Theory and its Ramifications was founded. This was purely devoted to knot theory.

The fundamental problem in Knot Theory was to distinguish knots from each other. Research is mainly done to solve this issue. In the mean time, interdisciplinary researches are gaining importance. With the discovery of the helical duplex structure of the DNA molecule, study of knotting and entangling during DNA synthesis is becoming essential. Polymer chemists began studying the knotting of long chain polymers. The chirality of molecules could also be studied using knot theory. The theory of knots also has a role to play in many other fields like Statistical mechanics, fluid mechanics, quantum theory etc. As a sub branch of topology, Knot Theory, connected with abstract algebra and graph theory is gaining great attention in the field of research.

The next chapter introduces some basic concepts of mathematical knots.

Chapter - 2

KNOT THEORY- SOME BASIC CONCEPTS

§. 2.1. Introduction

The simplest common knots which can be tied using a string are the overhand knot and the figure- eight knot.

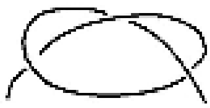


Figure 2.1 (overhand knot)

Figure 2.2 (figure- eight knot)

If the ends of the string are glued together then it forms a loop. Such a loop is a ***Mathematical knot***.

Figure 2.3 (trefoil knot)

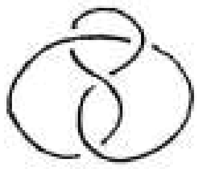


Figure 2.4 (figure- eight knot)

Thus a knot is a simple closed polygonal curve in space without any starting or ending point.

Definition 2.1.1

A subset K of R^3 is a ***knot*** if there exists a homeomorphism of unit circle C into three dimensional space R^3whose image is K.

C =$\{(x, y) \in R^3 : x^2 + y^2 = 1\}$.

The knot shown in figure 2.3 is called the ***trefoil knot*** or the ***clover–leaf knot***. The knot in figure 2.4 is the ***figure-eight knot*** or the ***Listing's knot***.

If the ends of a string are just glued together without knotting the string, then the knot obtained is called a ***trivial knot*** or an ***unknot***.

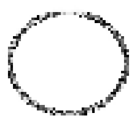


Figure 2.5 (the unknot)

If a knot is the union of a finite number of straight line segments called *edges* whose endpoints are called *vertices*, then the knot is called a ***polygonal knot***.

Definition 2.1.2

A knot can be diagrammatically represented using a projection. Consider a parallel projection $p: R^3 \to R^3$given by $p(x, y, z) = (x, y, 0)$. Let K be a knot and $\widehat{K}$ be the projection of K, that is, $p(K) = \widehat{K}$.

$\widehat{K}$ is called the ***regular projection*** of K if it satisfies the following conditions.

1. K has at most finite number of intersections.
2. If Q is any point of intersection of $\widehat{K}$, the inverse image $p^{-1}(Q) \cap K$, of Q in K has exactly two points, that is, Q is a *double point* of $\widehat{K}$.
3. A vertex of K (K being considered as a polygon) is not mapped onto a double point.

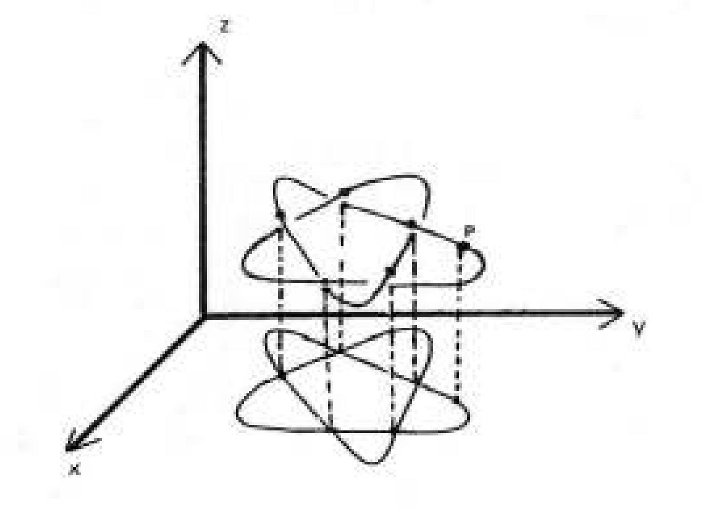


Figure 2.6 (Projection of a knot)

At a double point (or a crossing) of a projection the knot passes over or under itself. To show an overcrossing and an undercrossing, the projection close to the double point is drawn so that it appears to have a cut as shown in the figure 2.3 and 2.4. Such a diagram is called a ***regular diagram***. There may be more than one regular diagram for a particular knot as seen in the figure 2.7.

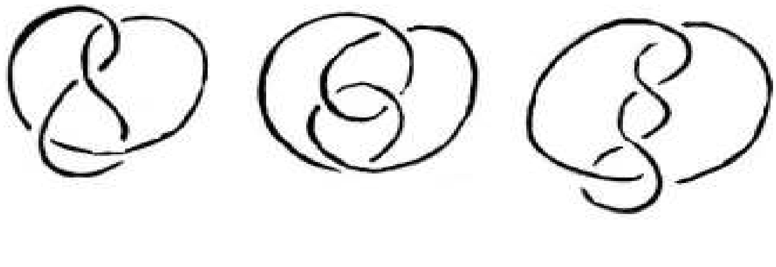


Figure 2.7 (Three regular diagrams of figure- eight knot)

Let an arbitrary point P on a regular diagram D of a knot K move along a fixed direction on D. If it moves alternatively from a segment that overpasses to a segment that underpasses at the crossings then the diagram is called an ***alternating diagram***. A knot with an alternating diagram is called an ***alternating knot***. The trefoil knot and the figure- eight knot are alternating knots (figure 2.3 and 2.4).

Composition of knots

Given any two projections of two knots J and K, a new knot projection is obtained by removing a small arc from each knot projection and then connecting the four endpoints by two new arcs in such a way that no new crossing is formed. The resulting knot is the ***composition*** (or ***connected sum***) of the two knots J and K and is denoted by **J ♯ K**.

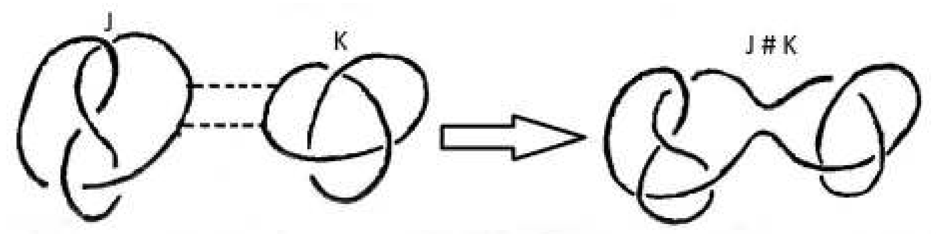


Figure 2.8 (A composition of figure- eight and trefoil knot.)

A knot is called a ***composite knot*** if it can be expressed as the composition of two non trivial knots. The knots that make the composite knots are called the ***factor knots***. If a knot is not the composition of two non trivial knots then it is called a ***prime knot***. The trefoil and figure- eight knot are prime knots.

A knot is a simple closed curve without a starting or an ending point. So an *orientation* can be assigned to a knot by choosing a direction to travel along the knot. The orientation is denoted by an arrow on the projection of the knot. Such a knot is called an ***oriented knot***. There are two possible orientations for each knot as shown in the figure 2.9.

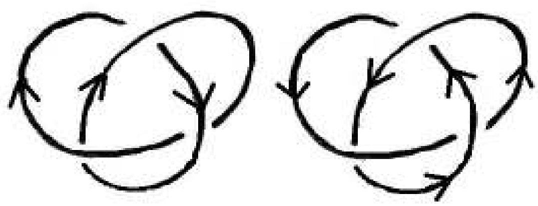


Figure 2.9 (oriented knots)

§. 2.2. Equivalence of knots

A knot made out of a string can be deformed in space without cutting open the closed knot. The deformed knot is considered to be the same as the original knot.

Definition 2.2.1

Two knots K_1 and K_2 are said to be ***equivalent*** if there exists a homeomorphism of R^3 onto itself which maps K_1 onto K_2. If two knots are equivalent then they are said to be of the ***same type***. Any knot equivalent to an unknot is of ***trivial type***. If a knot is equivalent to a polygonal knot then it is said to be a ***tame***; otherwise a ***wild knot***.

Any knot having a single crossing must be trivial. For, a one crossing knot can have only one of the four projections.

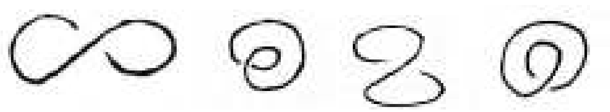


Figure 2.10 (one crossing knots)

The crossing can be untwisted to make it an unknot. Thus a non trivial knot contains more than one crossing in a projection.
There are also no two crossing nontrivial knots. For, the possible diagrams for a two crossing knot are the following.

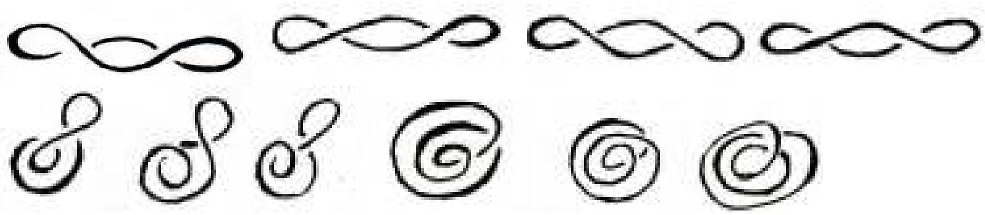


Figure 2.11 (2-crossing knots)

The two crossings can be untwisted to get a trivial knot.

A trefoil knot is a nontrivial knot. It cannot be deformed into an unknot without cutting it open. Thus a nontrivial knot will have at least three crossings.

Definition 2.2.2

If an *orientation* is assigned to the knots, then two knots K_1 and K_2 are ***equivalent*** if there exists an orientation preserving homeomorphism of R^3 that maps K_1 onto K_2. If K_1 is equivalent to K_2 it is denoted by $\boldsymbol{K_1 \approx K_2}$.

Let K be a knot which is assigned an orientation and let –K denote the same knot with opposite orientation. If K and –K are equivalent then K is said to be ***invertible***.

Example 2.2.1

Consider $\varphi(x,y,z) = (-x,-y,z)$, a $180°$ rotation about the z- axis.This is an orientation preserving auto- homeomorphism of R^3. If K_1 is a trefoil knot oriented clockwise and K_2 a trefoil knot oriented anticlockwise, then φ maps K_1 onto K_2 . Thus K_1 is equivalent to K_2 which has a reverse orientation and hence the trefoil knot is invertible.

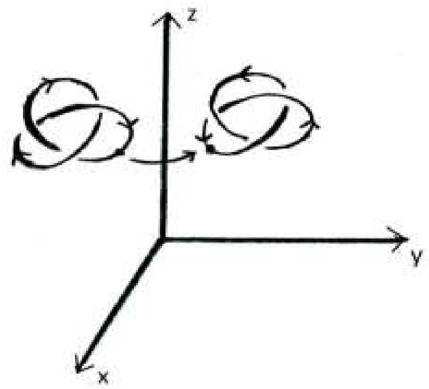


Figure 2.12 (K_1 mapped to K_2 which has reverse orientation)

The knot shown in figure 2.13 with 8 crossings is not invertible.

Figure 2.13 (not invertible)

If this knot is composed with itself in two different ways, then two distinct knots which are not equivalent are obtained.

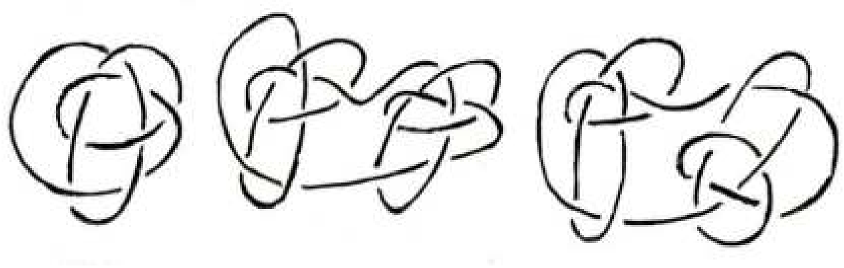


Figure 2.14 (two different knots composed in different ways)

Theorem 2.2.1(The uniqueness and existence of a decomposition of knots)

1. Any knot can be decomposed into a finite number of knots.
2. This decomposition, excluding the order, is unique. That is, if a knot is decomposed in two ways as $K_1K_2K_3 \dots K_m$ and $K'_1K'_2K'_3 \dots K'_n$, then n=m. Also, renumbering the subscript suitably of $K_1, K_2, K_3, \dots, K_m$, $K_1 \approx K_1'$, $K_2 \approx K_2'$, ... $K_m \approx K_m'$. ●

Proposition 2.2.1

$K_1 \sharp K_2$ is equivalent to $K_2 \sharp K_1$, with orientation, that is, commutative law holds for sum of two knots.
Also, associative law holds, that is, $K_1 \sharp (K_2 \sharp K_3) \approx (K_1 \sharp K_2) \sharp K_3$. ●

Let $\mathcal{A}$ denote the set of all oriented knots and let sum of knots be defined on $\mathcal{A}$ as the connected sum. Then $\mathcal{A}$ is a semi- group but not a group under this operation. $\mathcal{A}$ contains the unit element, the trivial knot, O but it does not contain inverse elements. For example, for the oriented trefoil knot K there does not exist a knot K′ such that $K \sharp K' \approx O$.

To show the equivalence of knots there are some invariants which can be used.

§. 2.3. Classical Knot Invariants

Definition 2.3.1

Let K be a knot and let a specific quantity ρ (K) be assigned to it. Then ρ (K) is said to be a knot ***invariant*** if it is equal for any knot equivalent to K.

If a knot K is equivalent to another knot K′ then K′ can be obtained from K by the following operations called ***elementary knot moves*** a finite number of times. Here a knot is considered as a polygon.

1. An edge AB in space of K can be divided into edge AC and CB where C is a point on AB.

1′. If AC and CB are adjacent edges of K such that AB is a straight line then the point C can be removed.

2. Suppose C is a point in space that does not lie on K. If Δ ABC intersects K only on the edge AB then AB can be replaced by AC and CB.

2′. If ABC is a triangle in space such that AC and CB are adjacent edges of K and Δ ABC does not intersect at any other point then AC and CB can be deleted and AB can be added as an edge instead.

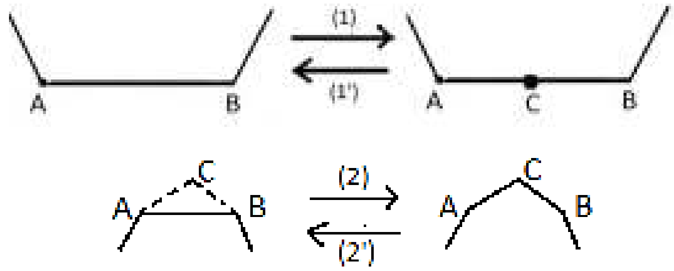


Figure 2.15 (elementary knot moves)

Thus suppose that a knot is made out of a string which is modelled by a knot diagram. The string can be rearranged in three dimensional space so that it does not pass through itself. Then the knot diagram may be different from the first. Such a rearrangement in three dimensional space is called ***an ambient isotopy***.

If the deformation of the knot is in the projection plane then the deformation is called a ***planar isotopy***.

Studying equivalent knots in space can be done by projecting them on a plane and studying the knot diagrams. If D and D′ are knot diagrams of K and K′ respectively, then K ≈ K′ if and only if D ≈ D′.

Reidemeister Moves

A **Reidemeister move** is one of the three ways to change a projection of the knot that will change the relation between the crossings.

I. First Reidemeister move

A twist can be put in or taken out in a knot.

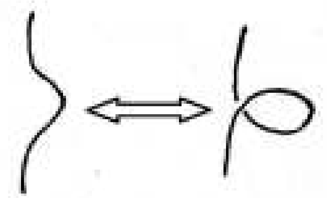


Figure 2.17 (First **Reidemeister move**)

II. Second Reidemeister move

Two crossings can be added or removed as in figure 2.18.

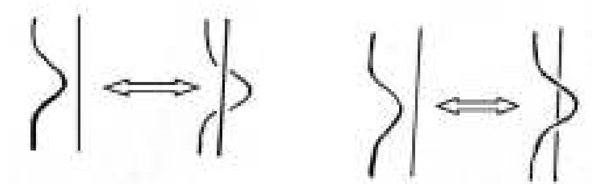


Figure 2.18 (Second Reidemeister move)

III. Third Reidemeister move

A strand of the knot can be slid from one side of the crossing to the other side as in figure 2.19.

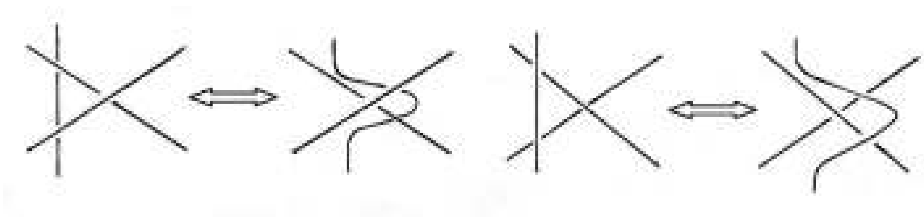


Figure 2.19 (Third Reidemeister move)

The German mathematician Kurt Reidemeister proved that if there are two projections of the same knot, then one can be obtained from the other by a series of **Reidemeister** moves and planar isotopies.

The following examples demonstrate this.

Example 2.3.1

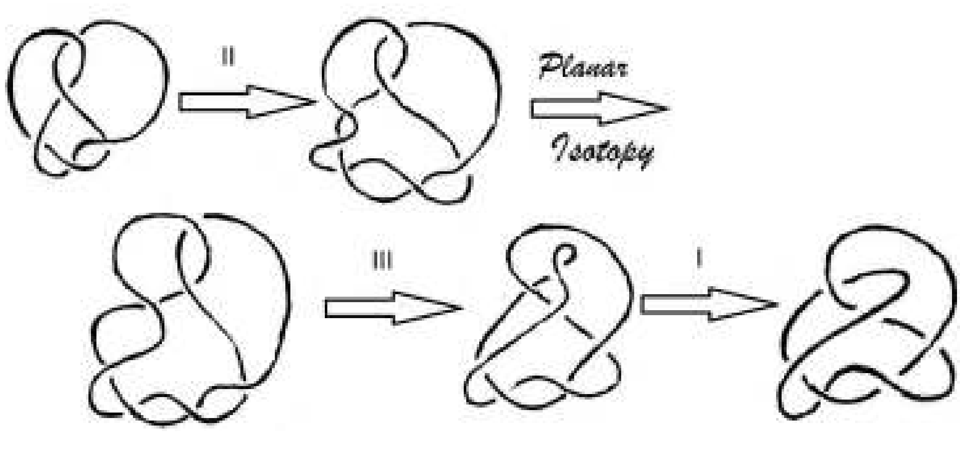


Figure 2.20 (Reidemeister moves to show the equivalence of two projections)

Example 2.3.2

The figure- eight knot is equivalent to its mirror image.

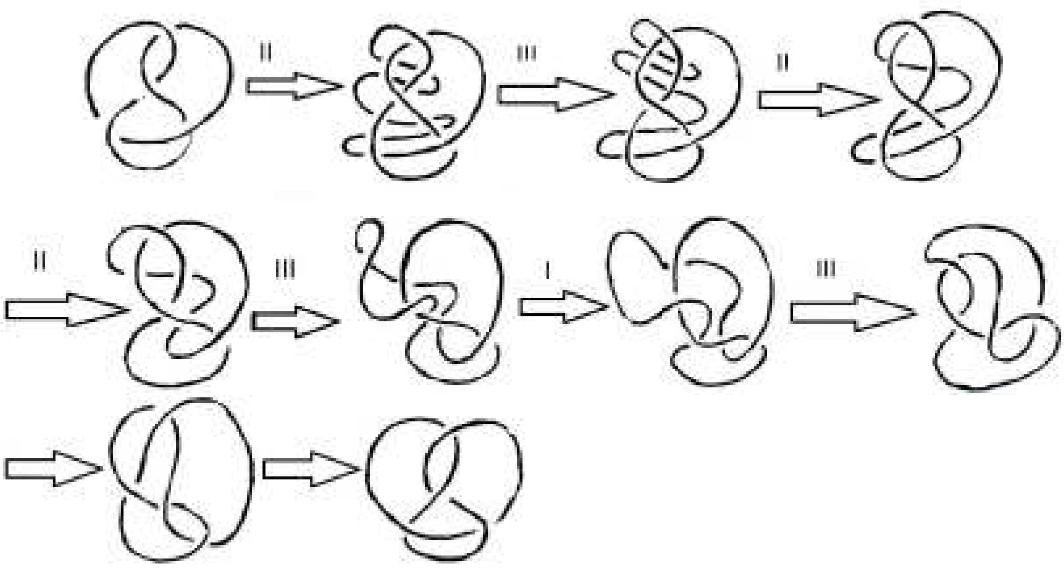


Figure 2.21 (Sequence of Reidemeister moves)

Example 2.3.3

The following two projections represent the same knot.

Figure 2.22

The series of Reidemeister moves in figure 2.23 shows the equivalence.

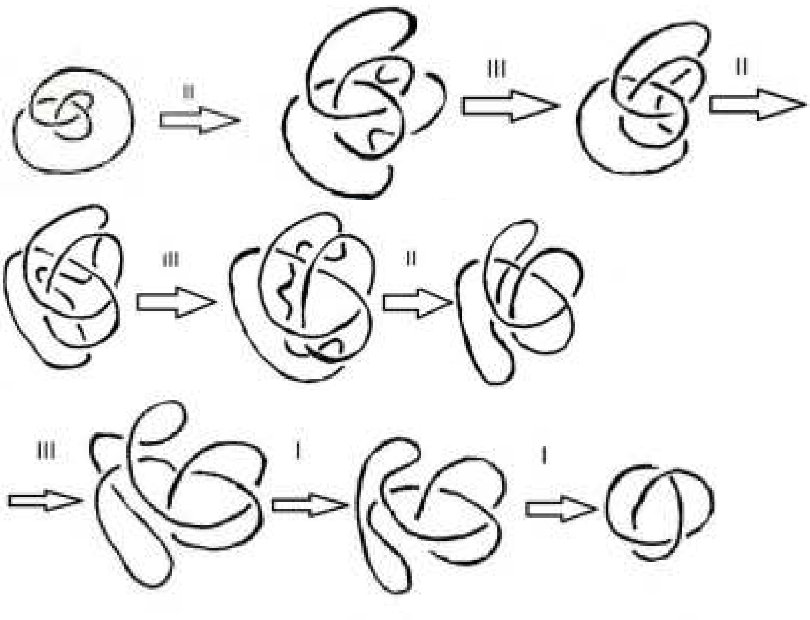


Figure 2.23

Example 2.3.4

The projection in figure 2.24 can be entangled into an unknot.

Figure 2.24

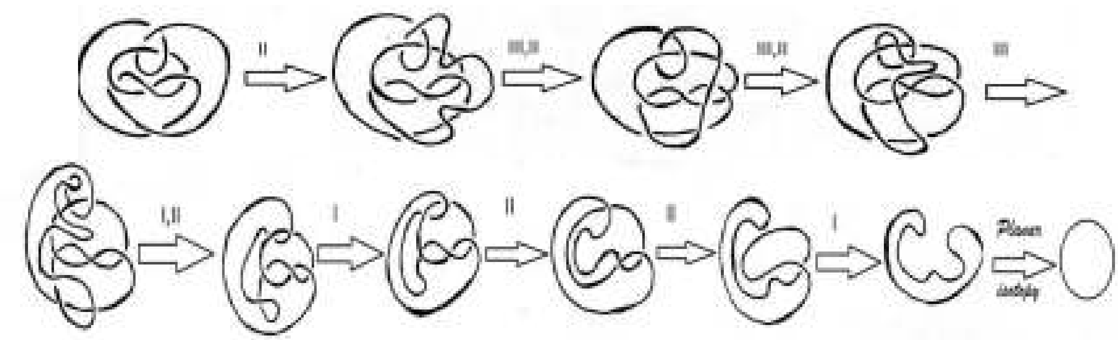


Figure 2.25 (sequence of Reidemeister moves)

As the number of crossings increases the number of Reidemeister moves required to get one projection from another increases and the process becomes difficult. Crossing number, unknotting number, linking number, and the colouring number are some other elementary invariants to show that two knots are equivalent.

<u>Note:</u> Example 2.3.2 shows that the figure- eight knot is equivalent to its mirror image. Such knots are called ***amphichiral knots***. The trefoil knot is not amphichiral knot. It is said to be ***chiral***.

<u>The crossing number</u>

A regular diagram D of a knot K has at most a finite number of crossing points. But number of crossing points of two diagrams of the same knot may be different. For example, figure 2.7 shows three regular diagrams of figure- eight knot. The first two diagrams have four crossing points. But the third one has five.

Definition 2.3.2

The minimum number of crossings points of all the regular diagrams of a knot K, denoted by *c(K)* is called the ***crossing number*** of K. A regular diagram of K that has *c (K)* crossing points is the ***minimum regular diagram*** of K.

Theorem 2.3.1

$c(K) = \min_{D\in\mathfrak{D}} c(D)$ is a knot invariant, where $\mathfrak{D}$ is the set of all regular diagrams D of K.

Proof:

Suppose that D_0 is the minimum regular diagram of a knot K. Let K′ be a knot equivalent to K and let D_0' be a minimum regular diagram of K′. Since D_0' is also a regular diagram of K, by definition of $c(K)$,

$$c(K) \leq c(D_0) \leq c(D_0') .$$

Similarly, D_0 is a regular diagram of K′ so that

$$c(K') \leq c(D_0') \leq c(D_0) .$$

Thus, $c(D_0) = c(D_0')$. Hence $c(D_0)$ is the minimum number of crossing points of the knots equivalent to K. Consequently, $c(K)$ is invariant. ●

Note: For a non- trivial knot K, $c(K) \geq 3$.

Example 2.3.5

The crossing number of a trefoil knot is 3 and that of figure- eight knot is 4. So they are not equivalent.

The Bridge Number

Let D be a knot diagram of a knot K. At each crossing point of D there is an overcrossing and an undercrossing. If $c(D) \geq 1$ then there is a unique positive integer k such that there is a finite sequence $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ of 2k points of D, each of which is neither an overcrossing nor an undercrossing of D, such that $[s_1, f_1]$, $[s_2, f_2]$, ..., $[s_{k-1}, f_{k-1}]$, $[s_k, f_k]$ and $[f_1, s_2]$, $[f_2, s_3]$, ..., $[f_{k-1}, s_k]$, $[f_k, s_1]$ are overpasses and underpasses of D respectively, with respect to $s_1, f_1, s_2, f_2, \dots, s_k, f_k$. Here $[s_i, f_i]$ for each $i \in \{1,2, \dots, k\}$ is the closed arc of D from s_i to f_i which contains atleast one over crossing but no under crossing and $[f_i, s_{i+1}]$ for each $i \in \{1,2, \dots, k\}$ is the closed arc of D from f_i to s_{i+1} which contains atleast one under crossing but has no over crossing. Such a sequence $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ is said to be an *over- underpass sequence* of D. Since any over-underpass sequence of D consists of 2k points, there are k overpasses (or underpasses). The number of overpasses (or underpasses), k, is called the *length of the over-underpass sequence* and is denoted by *b (D).* *b (D)* is also called the *bridge number* of D. If a knot diagram has no crossing then ***b (D)* = 0.**

Definition 2.3.3

For any knot K denote $b(K) = \min_{D\in\mathfrak{D}} b(D)$ where $\mathfrak{D}$ is the set of all regular diagrams D of K. Then the number $b(K)$ is called the ***bridge number*** of K (or the ***bridge index***).

Example 2.3.6

The bridge number of the trefoil knot as seen in figure 2.26 is b (D) = 3. The bridge number of the regular diagram in figure 2.27 is b (D) = 2.

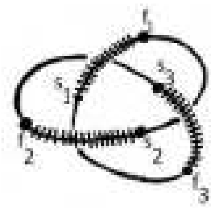


Figure 2.26

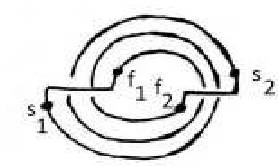


Figure 2.27

The regular diagrams in figures 2.26 and 2.27 represent a trefoil knot. From the above examples the bridge number of the trefoil knot must be 2.

<u>Note</u>: If the bridge number of a knot is 1, then it must be an unknot. For, if b (K) =1 then the knot K has only one overcrossing regular diagram, that is, K has one crossing point. But any knot with one crossing point is an unknot.

Theorem 2.3.2

Suppose K_1 and K_2 are two arbitrary knots. Then

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1. \qquad \bullet$$

If a knot K_1 is equivalent to a knot K_2 then b (K_1) = b (K_2). So the bridge number of a knot K, b (K) is an invariant for K.

Theorem 2.3.3

If D is a minimal diagram of a knot K with respect to crossings, then

$$b(D) \le c(D) \le b(D)(b(D) - 2). \qquad \bullet$$

Theorem 2.3.4

If K is a nontrivial knot and D is a minimal diagram of K respect to crossings, then

$$1 + \sqrt{1 + c(D)} \le b(D) \le c(D). \qquad \bullet$$

The Unknotting Number

Let *D* be a regular diagram of an arbitrary knot. *D* can be changed to the regular diagram of an unknot by exchanging the over and undercrossing segments at several crossing points of *D* and using necessary ***Reidemeister moves***. Such an operation of exchanging the over and undercrossing segments at a crossing point is called an ***unknotting operation***.

The ***unknotting number*** of *D* is defined as the minimum number of unknotting operations that are required to change *D* into the regular diagram of the Trivial Knot. The *Unknotting Number* of *D* is denoted by *u(D), u(D)* is not an invariant of *K* .

Example 2.3.7

The regular diagrams D_1 and D_2 in figure 2.28 below are equivalent.

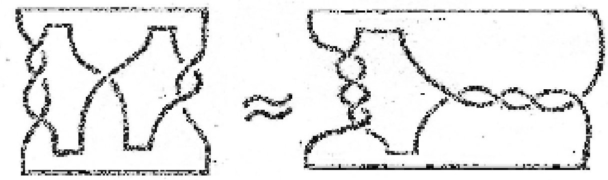


Figure 2.28

But $u(D_1)=1$ and $u(D_2)=2$

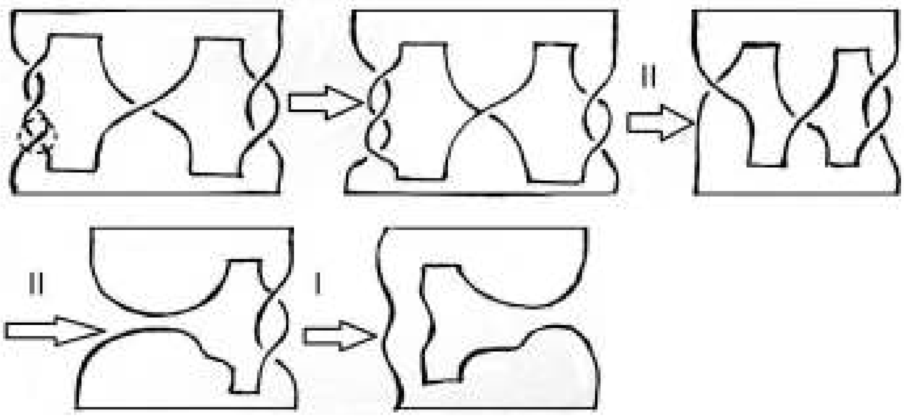


Figure 2.29a

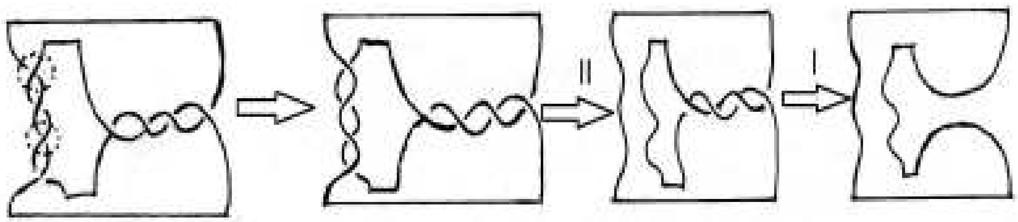


Figure 2.29b

Definition 2.3.4

If K is a knot, then the *Unknotting Number* of *K*, denoted by *u(K)* is

$$u(K) = \min_{D \in \mathfrak{D}} u(D)$$

where *D* is the set of all regular diagrams of *K* .

Theorem 2.3.5

If *K* is a knot, then *u(K)* is an invariant of *K*. ●

Links

Definition 2.3.5

A ***link*** is a finite ordered collection of knots that do not intersect each other. Each knot is said to be a ***component*** of the link.

Figure 2.30 shows two projections of the link known as *Whitehead Link*. It has two components. The link called the *Borrowmean Rings* in figure 2.31 has three components.

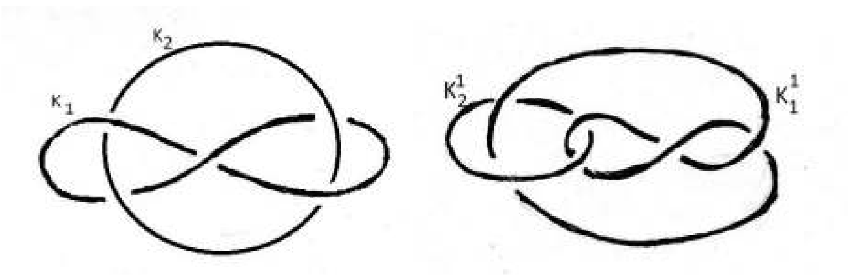
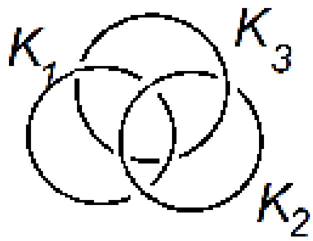


Figure 2.30

A knot can be considered to be a link with one component. The properties applied on knots can be applied on links as well.



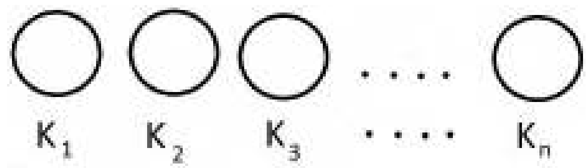


Figure 2.31

Figure 2.32

A trivial n- component link consists of n disjoint trivial knots (figure 2.32).

Two links L = $\{K_1, K_2, \ldots, K_m\}$ and L′ = $\{K'_1, K'_2, \ldots, K'_m\}$ are equivalent if the following two conditions hold.

(1) m = n, i.e. L and L′ have the same number of components.

(2) L can be changed to L′ by performing the elementary knot moves a finite number of times.

In other words, K_1 can be changed to K_1' , K_2 to K_2' ,..., K_m to K_n' where m = n.

In figure 2.30 $K_1 \approx K_1'$ and $K_2 \approx K_2'$ and so the two diagrams represent the same link. But 2-component trivial link and the Hopf Link are not equivalent though both contain 2 components and each component the Hopf link is equivalent to a component of the 2-component trivial link. In order to show their inequivalence the property splittability of a link is defined.

A link is ***splittable*** if the components of the link can be deformed so that they lie on different sides of a plane in three – space. This property can be used to show that the unlink (or the trivial link) with two components is different from the Hopf Link.

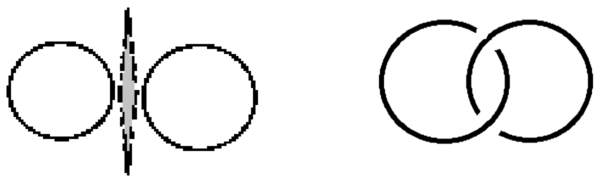


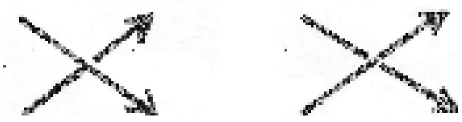
Figure 2.33 (2-component trivial link and the Hopf Link)

The Unlink is splittable whereas the two components of the Hopf Link cannot be separated by a plane. But a Whitehead link is a 2- component link which is not splittable and not equivalent to the Hopf link.

To show the equivalence of oriented links, the *linking number* can be used as an invariant.

At a crossing point, c, of an oriented regular diagram two possible configurations exists as in figure 2.34. Assign **sign(c) = +1** to the crossing point of type I and **sign(c) = -1** to that of type II. The crossing point of type I is said to be positive and that of type II is negative.

Note: If a crossing is of first type, then rotating the understrand clockwise lines it up with the overstrand so that their arrows match. If a crossing is of the second type, then rotating the understrand counterclockwise lines it up with the overstrand so that their arrows match.



sign(c) = +1 sign(c) = -1

Figure 2.34

Definition 2.3.6

Suppose that D is an oriented regular diagram of a two component link $L = \{K_1, K_2\}$. Also, suppose that the crossing points of D at which the projections of K_1 and K_2 intersect (self - intersections of each component not considered) be $C_1, C_2, \ldots, C_m$. Then $\frac{1}{2}\{\boldsymbol{sign(C_1) + sign(C_2) + \cdots + sign(C_m)}\}$ is called the ***linking number*** of K_1 and K_2 and is denoted by $\boldsymbol{lk\ (K_1, K_2)}$.

Theorem 2.3.6

The linking number $lk\ (K_1, K_2)$ is an invariant of the link $L = \{K_1, K_2\}$.

<u>Proof:</u>

Any projection of a link can be obtained from another by a sequence of *Reidemester moves*. So it is enough to prove that the *Reidemester moves* do not change the linking number.

The first *Reidemester move* can create or eliminate a self – crossing in one of the components. It will not affect the crossing involving both the components. Thus the first *Reidemester move* leaves the linking number unaltered.

To show the effect of Type II *Reidemester move*, consider two strands from different components with certain orientation as in figure 2.35. One of the new crossings contribute a +1 to the sum while the other a -1. Thus the net contribution to the linking number is 0. Even if the orientation of one strand is changed, the net effect is 0.

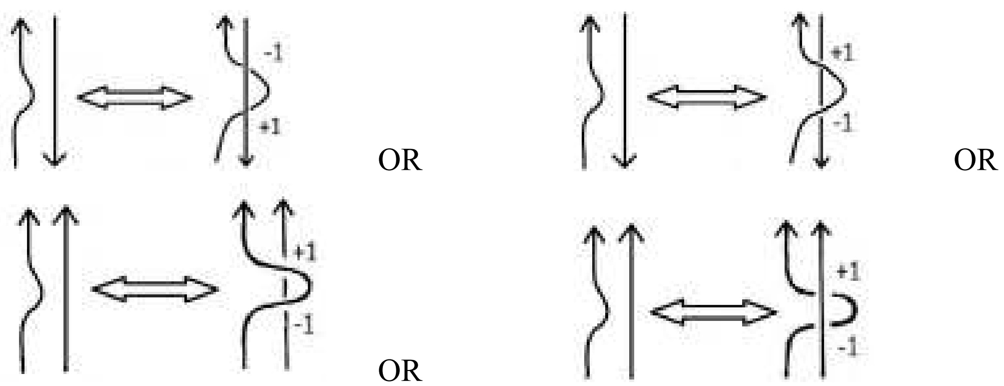
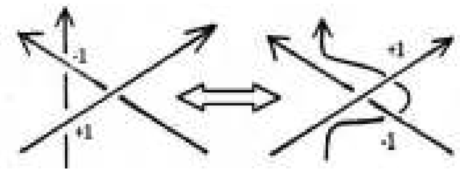


Figure 2.35

In the case of Type III move, once the orientation of the strands are chosen for each of the three strands, the +1 and the -1 assigned to the crossings will not change if the strand is slided over so the linking number is unchanged (figure 2.36)



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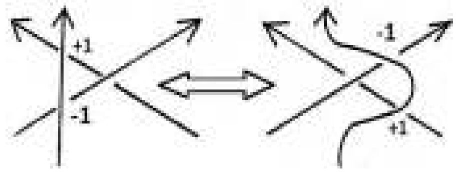


Figure 2.36

Hence the linking number is an invariant of oriented links. ●

Since the linking number is invariant for a link L, it is denoted by ***lk (L).***

Example 2.3.8

Consider the links L and L' in figure 2.37.

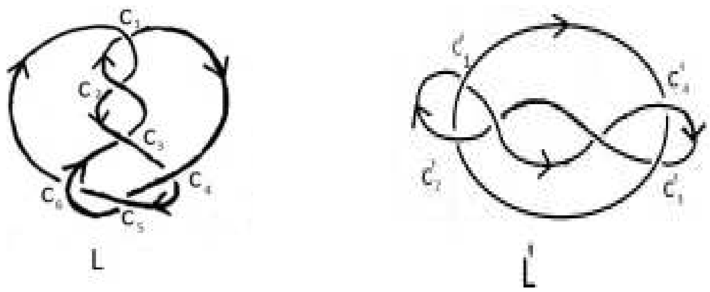


Figure 2.37

$lk\ (L) = \frac{1}{2}\{+1+1+1+1+1+1\} = 3$ $\quad$ $lk\ (L') = \frac{1}{2}\{-1-1-1-1\} = 2$

Since the linking numbers of L and L' are not equal they are different links.

Example 2.3.9

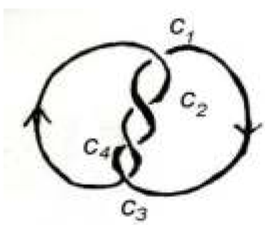


Figure 2.38

The linking number of the link in figure 38 is $\frac{1}{2}\{+1+1+1+1\}$=2.

Example 2.3.10

Compute the linking number of the link is figure 2.39. Now reverse the direction on one of the components and recompute it.

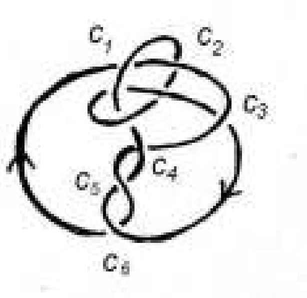


Figure 2.39

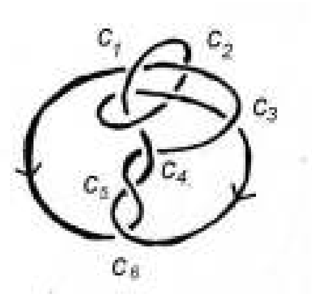


Figure 2.40

Linking number is equal to $\frac{1}{2}\{-1-1+1+1+1+1\}$=1.

Reversing the direction of one of the components as in figure 2.40 the linking number becomes $\frac{1}{2}\{+1+1-1-1-1-1\}$= -1

In the above example, if $-K_2$ denotes the knot K_2 with reversed orientation then

$lk(K_1, -K_2) = -lk(K_1, K_2)$. Thus linking number depends on the orientation chosen.

Example 2.3.11

The linking number of the *Whitehead Link* is zero

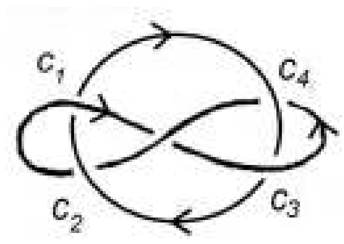


Figure 2.41

lk(L)=$\frac{1}{2}\{+1+1-1-1\}$=*0*

So the linking number of a Trivial 2 component link and the *Whitehead Link is* zero. But they are not equivalent.

Suppose L is a link with n components, say, L = $\{K_1, K_2, \dots, K_n\}$. The linking number of L is defined as an extension of the linking number of two components K_i and K_j , $lk\,(K_i, K_j)$, $1 \le i \le j \le n$, as $\sum_{1\le i\le j\le n} lk\,(K_i, K_j) = lk(L)$.

The Colouring Number of a Knot

A ***strand*** in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between.

A projection of a knot or link is ***tricolourable*** if each of the strands in the projection can be coloured one of three different colours so that at each crossing either 3 different colours come together or all the same colours come together. At least two of the colours must be used.

Example 2.3.12

The two projections of the trefoil knot is tricolourable as in figure 2.42.

Figure 2.42

Example 2.3.13

The six crossings knot projection in figure 2.43 is tricolourable. The projection of a seven crossings tricolourable knot is shown in figure 2.44.

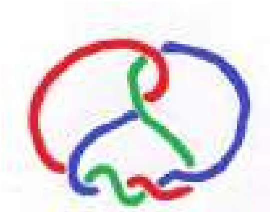


Figure 2.43

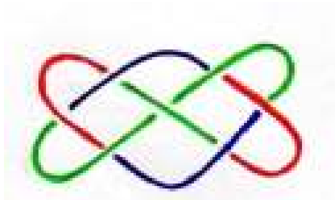


Figure 2.44

If a projection is tricolourable, then the *Reidemister moves* preserves tricolourability.

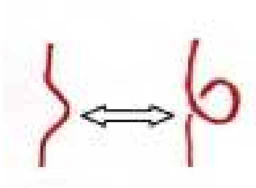
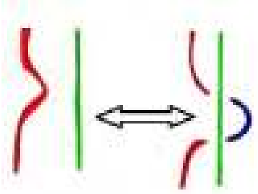


Figure 2.45

For, if a crossing is introduced or removed by Type I *Reidemister move* then leave all the strands involved the same colour and tricolourability will be preserved.

If two strands are of different colours and a Type II *Reidemister move* introduces two new crossings, then change the colour of the new strand to a third colour and the resulting knot projection becomes tricolourable. If the original strands are of the same colour, then leave the new strand the same colour.



OR

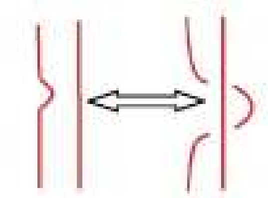
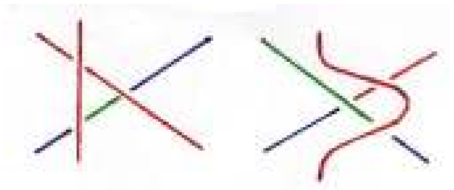


Figure 2.46

If a Type III *Reidemister move* is introduced tricolourability will be preserved as in figure 2.47.



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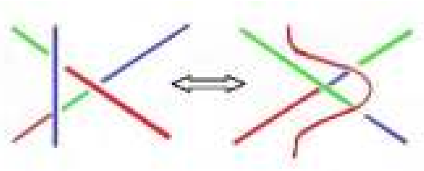

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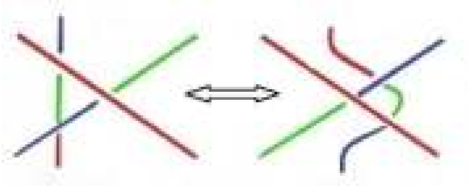


Figure 2.47

Tricolourability can be defined in the following manner to generalise it to p – colourable knots where p is prime.

Definition 2.3.7

Suppose that the projection $\widehat{K}$ of a knot K has n crossings $P_1, P_2, \dots, P_n$. Each crossing point P_i is the projection of two points P_i' and P_i'' of K. So these points divide K into 2n segments $A_1, A_2, \dots, A_{2n}$, say. To each of the segments assign one of the three colours red, green or blue in such a way that the following conditions are satisfied.

(i) If A_k and A_l are as in the figure 2.48 then assign the same colour to both.

(ii) Either assign A_k(orA_l), A_r and $A_s$the same colour or three different colours.

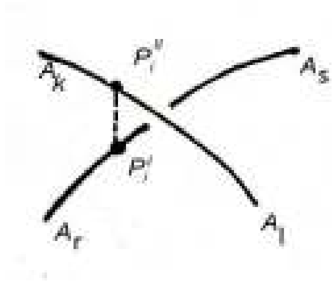


Figure 2.48

A regular diagram D of K which can have three colours assigned throughout the diagram in this manner is said to be ***tricolourable***.

Tricolourability can be extended to p – colourability where p is a prime number.

Definition 2.3.8

Consider the segments $A_1, A_2, \dots, A_{2n}$ of K and assigned to each segment A_i an integer λ_i such that $\lambda_i \in \{0, 1, \dots, p-1\}$. These λ_is are so assigned that the following conditions are satisfied with regard to figure 2.48.

(i) $\lambda_k = \lambda_l$

(ii) $\lambda_r + \lambda_s \equiv \lambda_k + \lambda_l (mod\ p)$

If $\lambda_1 = \lambda_2 = \dots = \lambda_{2n} = 0$, then all the segments have the same colour and the colouring is said to be a **trivial colouring** of a regular diagram D.

A regular diagram D is said to be ***p – colourable*** if at least two segments are assigned two different integers.

As in the case of tricolourability if a knot (or link) has at least one p – colourable a regular diagram, then every regular diagram is p – colourable.

Example 2.3.14

Show that the figure-eight knot is 5 – colourable but not 3 colourable.

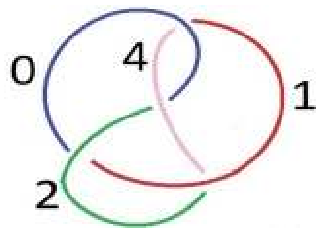


Figure 2.49

The possible choices of λ_r, λ_s and λ_k are (1, 4, 0), (4,1,0), (2,3,0), (3,2,0), (0,2,1), (1,1,1), (2,0,1), (0,4,2), (2,2,2), (4,0,2), (0,1,3), (1,0,3), (0,3,4), (1,2,4), (2,1,4), (3,0,4) where $\lambda_i \in \{0,1,2,3,4\}$. Thus choosing values from this set it can be shown that figure-eight knot is 5-colourable (figure 2.49).

If $\lambda_i \in \{0,1,2\}$ the possible values λ_r , λ_s and λ_k can be taken so that the required condition is satisfied are (1,2,0), (2,1,0), (1,1,1,), (0,2,1), (2,0,1), (0,1,2), (1,0,2) and (2,2,2).

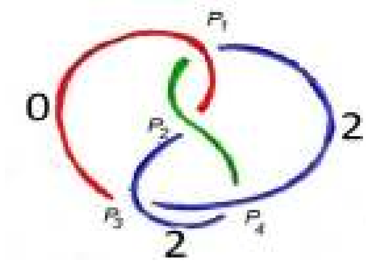


Figure 2.50

Suppose the overstrand through the crossing point at P_1 is assigned $\lambda_k = \lambda_l = 0$. Then at this point $\lambda_r = 1, \lambda_s = 2$ or vice versa. Let the overstrand at the crossing point P_2 be assigned 1 and that at P_4 be 2. At P_2, λ_r and λ_s can take values 0 and 2 only. But this assignment is not possible since the overstrand at P_4 is assigned 2. By a similar argument other possibilities are also ruled out. Hence figure 8 knot is not tricolourable. But it is 5 – colourable.

The above example shows that figure-eight knot is different from the trefoil knot since the trefoil knot is tricolourable but figure-eight knot is not. Also, since figure-eight knot is 5-colourable, it is not an unknot. It is a nontrivial knot.

All the invariants mentioned above, that is, the crossing number, the unknotting number, the linking number or the p-colourability show only the inequivalence of two knots. If any of these are the same for two knots it cannot be concluded that the two knots are equivalent.

Chapter- 3

TABULATING KNOTS

The first work on tabulating knot projections was done in the 1880s by the Rev.Thomas P Kirkman. A Scottish Physicist. Peter Guthrie Tait applied the ideas of Kirkman to tabulate all the alternating knots up to 10 crossings. C N Little, a professor at the University of Nebraska, published a table of 43 non alternating knots of 10 crossings. Kenneth A Perko, a mathematician of New York, discovered duplication in this table .The two projections corresponding to the same knot came to be known as the Perko pair.

Figure 3.1 (the Perko pair)

In 1917, Mary G. Haseman listed all amphichiral knots of 12 crossings. Kurt Reidemeister classified knots up to 9 crossings in 1932. In 1969, John H. Conway invented a notation to denote knots. He used it to determine all prime knots of 11 or few crossings and all the nonsplittable prime links of 10 or fewer crossings.

In 1978, Alain Caudron published the first correct list of all prime knots up to 11 crossings after correcting the errors in Conway's table. Hugh Dowker invented a new notation for knots which was used by Morwen Thistlethwaite to develop an algorithm to generate knots on the computer. A list of knots up to 16 crossings was put forward by his team.

A list of knots and links is given at the end of the chapter.

§. 3.1. The Dowker Notation for Knots

Choose an orientation for the knot given by placing coherently directed arrows along the knot. Pick any crossing and label it 1. Leaving that crossing along the understrand in the direction of the orientation label the next crossing as 2. Continue to label the crossing with the integers in sequence until the knot has been traversed around once. Each crossing will have 2 labels on it as the knot passes through each crossing twice. Each crossing has one even number and one odd number labelling it. Thus there is a pairing between odd numbers and even numbers.

Example 3.1.1

In figure 3.2, each crossing is labelled by numbers from 1 to 18. The odd and even numbers are paired in the figure as

1	3	5	7	9	11	13	15	17
14	12	10	2	18	16	8	6	4

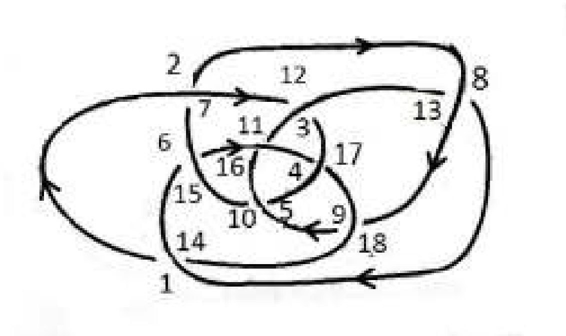


Figure 3.2

In short it can be written as 14 12 10 2 18 16 8 6 4. Thus from a projection of a knot a sequence of even integers are obtained where the number of even integers is exactly the number of crossings in the knot.

Example 3.1.2

Find a sequence of even integers that represents the projection of the knots 6_2 and 6_3 . How about a second sequence of even integers that also represents the same projection of 6_3 .

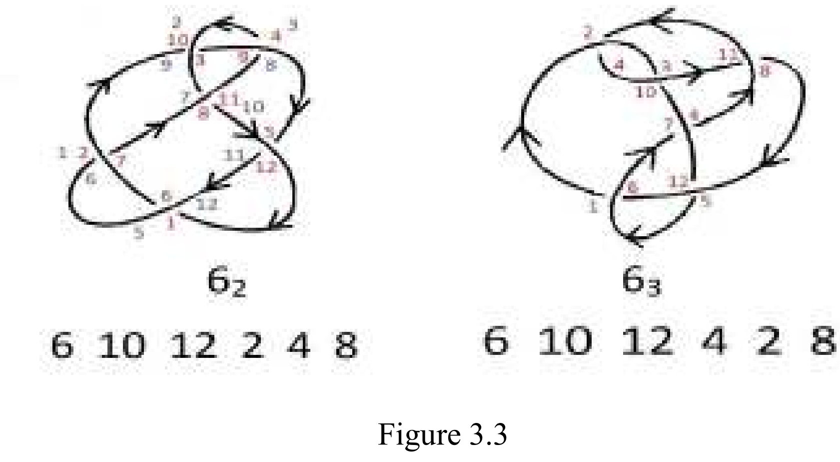


Figure 3.3

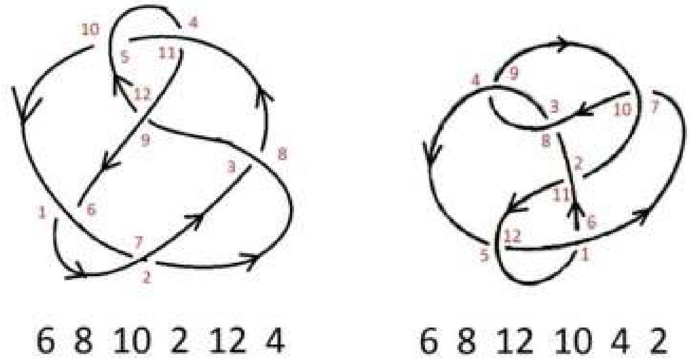


Figure 3.4

Example 3.1.3

Given a sequence of even integers that represents a projection of an alternating knot draw the projection, say, of sequence 8 10 12 2 14 6 4.

1	3	5	7	9	11	13
8	10	12	2	14	6	4

Start drawing the first crossing labelling it with a 1 and 8. Extend the understrand of the knot and then draw the next crossing which corresponds to 2. Label the crossing with a 2 and 7. Since the knot is alternating the strand goes over this crossing. Continue this process until the next integer that should be placed on a crossing already labelled on existing crossing. The knot then circles round to pass that crossing.

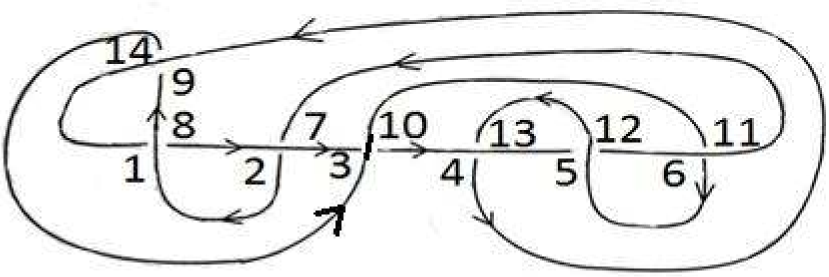


Figure 3.5

This is the 7-crossing knot 7_3 , as in the figure 3.6.

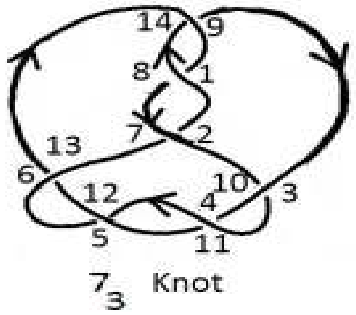


Figure 3.6

Example 3.1.4

Draw a picture of the projection of an alternating knot corresponding to the sequence 10 12 8 14 16 4 2 6

1	3	5	7	9	11	13	15
10	12	8	14	16	4	2	6

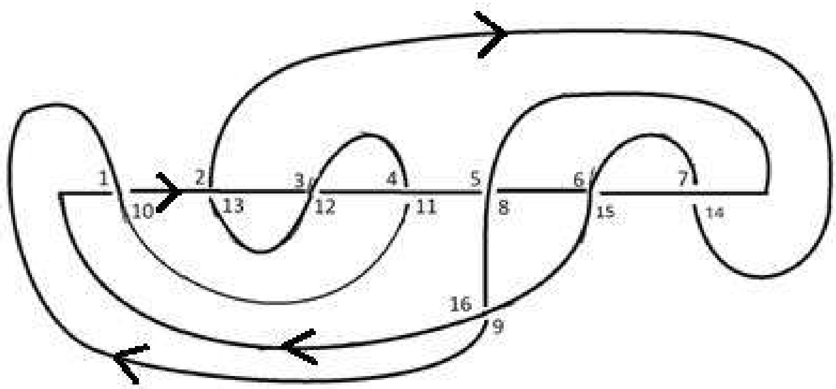


Figure 3.7

Example 3.1.5

Two knots can have the same Dowker Notation. For instance, the sequence 4 6 2 10 12 8 represents 2 distinct knots shown in the figure 3.8.

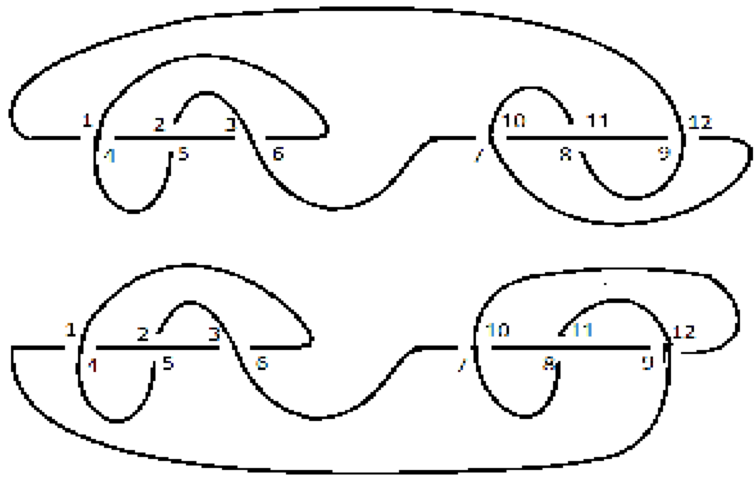


Figure 3.8

The 2 knots are composite knots. The sequence 4 6 2 10 12 8 is actually a shuffling of the 3 numbers 2,4,6 and then a shuffling of the 3 numbers 8,10 and 12

When the permutation of even numbers can be broken into 2 separate sub-permutations the resulting knots are composite and the knot is not completely determined by the Dowker Notation. However, if the sequence of even numbers cannot be split into sub-permutations either a particular knot or its mirror image results. When the knot is amphicheiral, only one knot can be the result.

A knot and its mirror image given by 8 6 10 2 4 as in figure 3.9, are distinct as projections on the plane but are equivalent projections on the sphere.

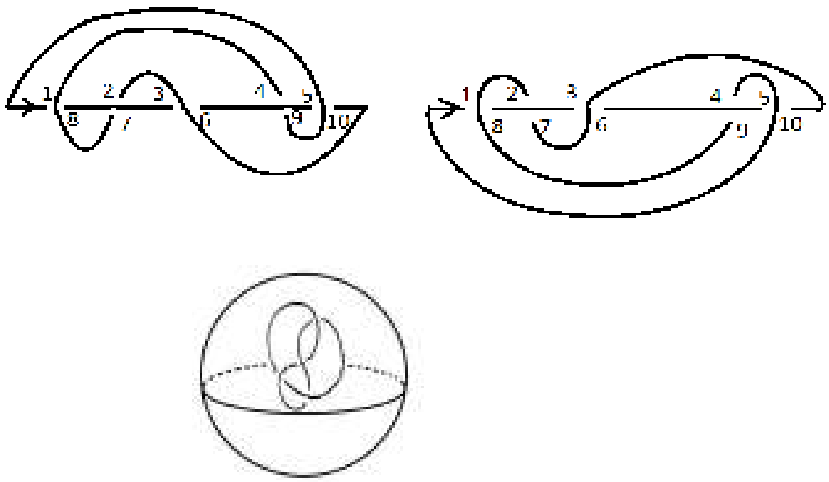


Figure 3.9

Example 3.1.6

Draw two projections given by 12 2 14 6 4 8 which are inequivalent as projections in the plane which are equivalent as projections on the sphere.

1	3	5	7	9	11	13
10	12	2	14	6	4	8

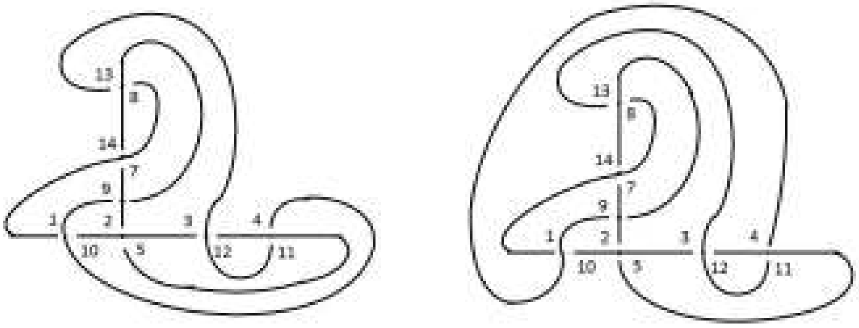


Figure 3.10

<u>**Dowker Notation for Non Alternating Knots**</u>

While traversing the knot using the labelling system described, an even integer and an odd integer is assigned at each crossing. If the even integer is assigned to the crossing while at the overstrand at that crossing, then the even integer is positive. But if the even integer is assigned to the crossing while at the understrand of the crossing then the corresponding even integer is negative.

Example 3.1.7

The Dowker notation for the knot in figure 3.11 is 6 -14 16 -12 2 -4 -8 10.

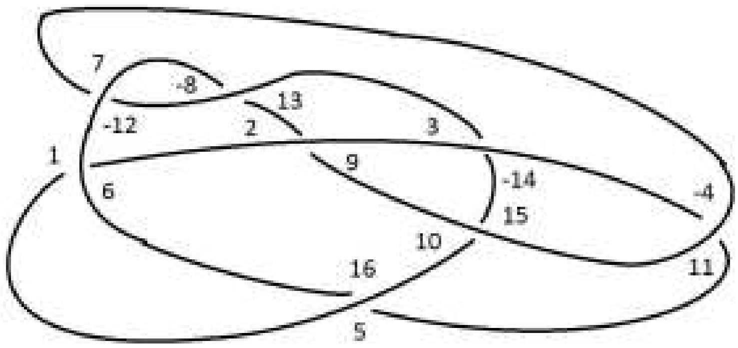


Figure 3.11

Example 3.1.8

Draw a projection of the knot corresponding to the sequence 14 12 -16 2 18 6 8 10 -4.

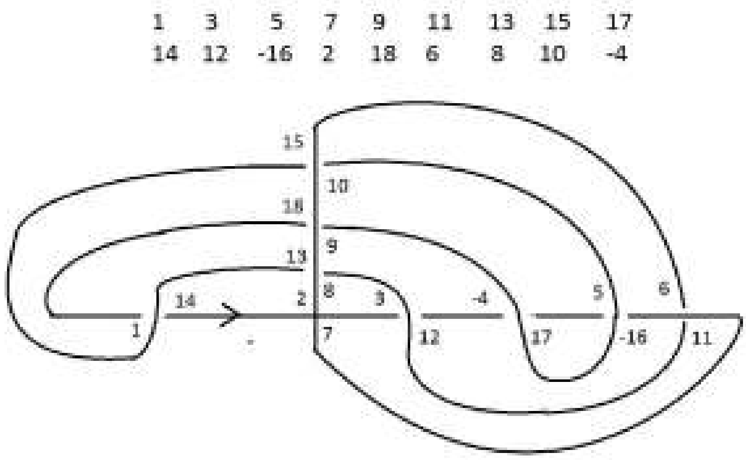


Figure 3.12

The number of sequence of the 14 numbers is 14! And putting of a +1 or -1 in front of each of the even numbers gives another factor 2^{14}. There aren't so many different knots with 14 crossings. Many of the sequences represent the same knot and may become projection of the same knot.

Morwen Thisllethwaite used the Dowker Notation to list all of the prime knots of 13 or fewer crossings.

§. 3.2. Conway's Notation

In 1969, John H. Conway introduced a notation for knots using what is called *tangles*. He used this notation to tabulate prime knots up to 11 crossing, prime links up to 10 crossings. This link is being applied to study the knotting of the DNA.

Suppose that 4 points are fixed on a unit sphere S^2, say, the 4 compass directions $NE = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $NW = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,

$SE = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ *and* $SW = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

These four points can be considered to lie on yz-plane. Consider two simple polygonal curves inside S^2not interesting each other with end points at the 4 points. This forms a tangle. If this tangle is projected onto the yz-plane, then a regular diagram of the tangle is obtained as in figure 3.14.

The tangles are used as the building blocks of knot and link projections. Joining the NW and NE points, the SW and SE points of a tangle T by simple arcs which lie outside S^2 a knot is obtained. Such a knot is called the ***numerator*** denoted by **N (T)**.

If the NW and SW points, the NE and SE points are connected instead the knot obtained is called the ***denominator*** denoted by **D (T)**.

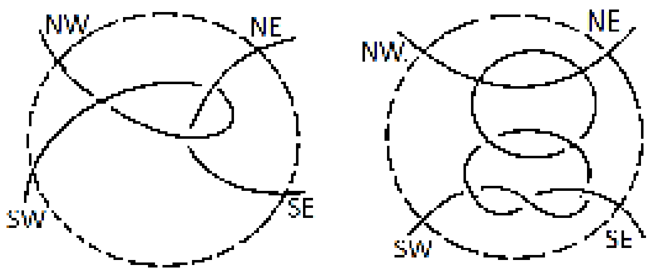
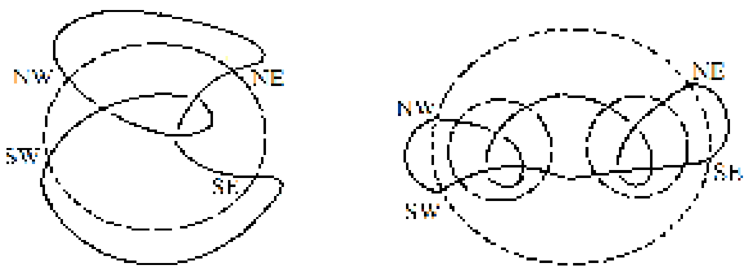


Figure 3.14



N (T) D(T)

Figure 3.15 (Numerator and denominator of a tangle)

Two tangles are said to be ***equivalent*** if one can be got from the other by a sequence of Reidemeister moves while the 4 ends of the strings in the tangle remain fixed and the strings of the tangle never travel outside the circle defining the tangle. The tangles in figure 3.16 are equivalent.

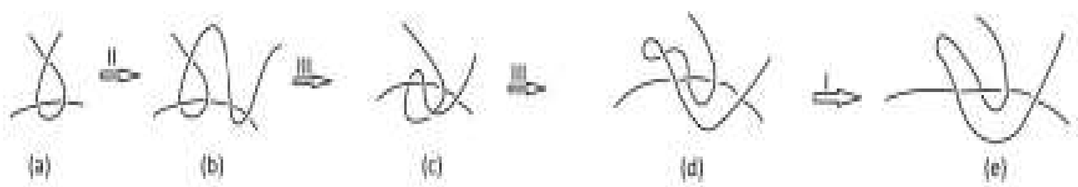


Figure 3.16 (sequence of Reidemeister moves showing equivalence of tangles)

Equivalence of tangles can be defined in a different way.

Here B^3 is the ball with S^2 as the boundary and the tangles can be considered to lie in B^3 meeting S^2 only at the 4 points.

Definition 3.2.1

Two tangles T_1 and T_2 are said to be equivalent if there exists an orientation preserving auto-homeomorphism $\varphi: B^3 \to B^3$ such that φ leaves S^2 fixed and $\varphi(T_1) = T_2$.

The simplest tangles are the $\infty -$ tangle and the 0- tangle shown in figure 3.17 (a), (b) respectively.

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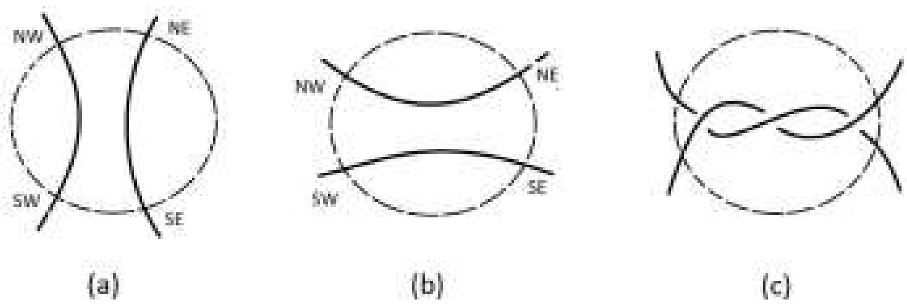


Figure 3.17

Definition 3.2.2

Suppose that there exists a homeomorphism of B^3 onto itself such that the set of 4 points $\{NE, NW, SE, SW\}$ is mapped onto itself (not necessarily identity map). Then a tangle which is the image of the $\infty -$ tangle under this homeomorphism is called a ***trivial tangle*** or a ***rational tangle***.

Example 3.2.1

If R^3 is rotated about the *x*-axis through an angle $\pi/2$, the image of the $\infty -$ tangle is a 0-tangle.Thus a 0-tangle is a rational tangle.

Suppose S^2 is rotated about the *z*-axis, keeping the northern hemisphere and the south pole fixed. Then the southern hemisphere is given a twist such that SW and SE exchange position. Such a twist is called a ***vertical twist***.

If S^2 is rotated about the *y*-axis keeping the western hemisphere and the point (0, 1, 0) on the equator fixed, then the eastern hemisphere is given a twist such that NE and SE exchange their position. Such a twist is called the ***horizontal twist***.

In a vertical twist if the twist is a right hand twist then is assigned a positive sign. In a horizontal twist a left hand twist is positive. In figure 3.17(c) two horizontal strings are given left handed twists three times and so the tangle is denoted by 3. If twists were right handed, the resulting tangle is denoted by -3.

<u>Note</u>: For a positive integer twist the over strand always has a positive slop.

More complicated tangles can be formed starting from the 3 tangles.

First reflect through the NW and SE diagonal line. Now wind the right hand ends of the tangle around each other. This tangle is denoted by 3 2, as the original tangle had 3 twists of the horizontal strings followed by a reflection and then 2 twists of the horizontal strings.

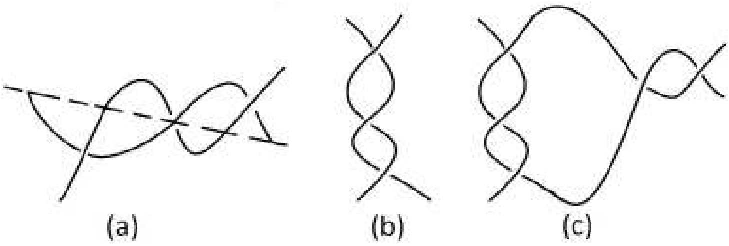


Figure 3.18 (the construction of 3 2 tangle)

<u>Note</u>: For a positive integer twist, the over strand has a positive slope even after reflection.

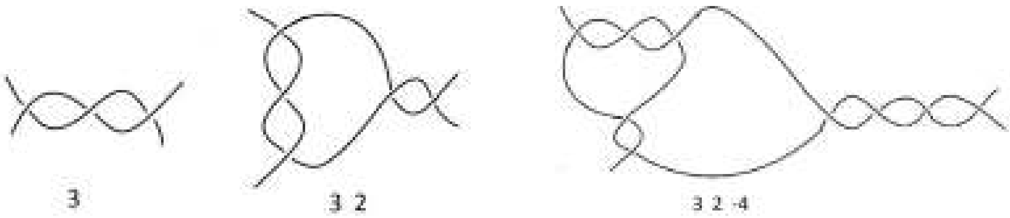


Figure 3.19

Figure 3.20

Any tangle constructed in this manner is a *Rational Tangle*. In general, a rational tangle can be obtained from a finite number of vertical and horizontal twists starting from an ∞-tangle or a 0-tangle.

If the rational tangle is represented by an even number of integers, say, $(a_1, a_2, \dots, a_n)$, then it can be constructed by simply starting with an ∞-tangle and performing, a_1 twists, followed by a_2 horizontal twists, and so on, ending with a_n horizontal twists. If n is odd, the tangle can be obtained by first performing a_1

number of horizontal twists on a 0-tangle, then a_2 vertical twists, and so on, ending up in a_n horizontal twists.

<u>Note</u>: If all the a_is are of the same sign then the regular diagram is an alternating diagram.

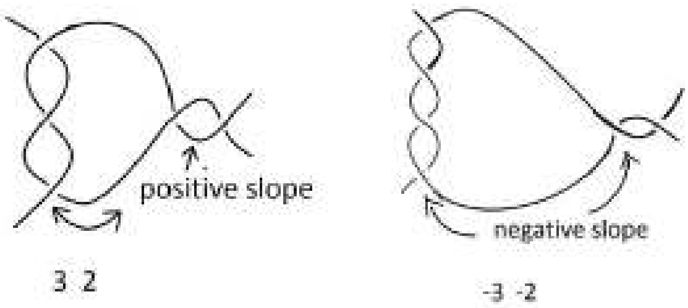
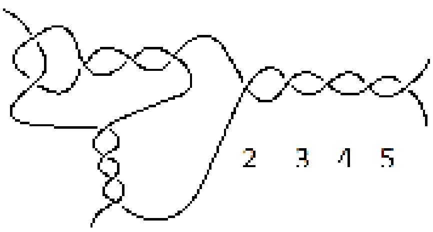


Figure 3.21

Example 3.2.2

The rational tangles corresponding to 2 -3 4 5 and 3 -1 3 -3 2 is shown in figure 3.22 and 3.23, respectively.



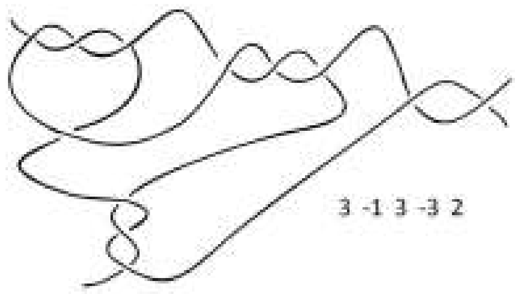


Figure 3.22 Figure 3.23

To show the equivalence of rational tangles, Conway used the concept of continued fractions in number theory.

Any real number can be expressed as a *continued fraction.*

Consider the continued fraction

$$2+\frac{3}{4+\frac{1}{5}} = 2+\frac{3}{{21}/{5}} = 2+\frac{5}{7} = \frac{19}{7}.$$

Thus $2+\frac{3}{4+\frac{1}{5}}$ is the continued fraction of the rational number $\frac{19}{7}$.

Now $\frac{19}{7} = 2 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}$ is another way of expressing $\frac{19}{7}$ as continued fractions. Here except for the initial integer the numerator of all fractions is 1. This continued fraction can be represented by a simple notation as [2, 1, 2, 2]. Also,

$\frac{19}{7} = 3 + \frac{1}{-3+\frac{1}{-2}}$ = [3, -3, -2].

Thus a rational number can be expressed in more than one different form of continued fractions.

Any rational tangle $a_1\ a_2 \ldots a_n$ where $a_i \neq 0$ corresponds to the fraction $\frac{p}{q}$ that has the continued fraction $[a_n, a_{n-1}, \ldots, a_2, a_1] = \frac{p}{q}$. This number is called the ***fraction of the tangle***. The fraction of the 0-tangle is 0 and that of the ∞-tangle is $0 + \frac{1}{0} = \infty$.

The following theorem gives the correspondence between rational numbers (including∞) and rational tangles.

Theorem 3.2.1

There is a 1-1 correspondence between the set of rational numbers (including∞) and the equivalence classes of the trivial tangles. In other words, if the trivial tangle $a_1\ a_2 \ldots a_n$ and $b_1\ b_2 \ldots b_m$ are equivalent, then their respective fractions that are expressed by corresponding continued fraction expressions $[a_n, a_{n-1}, \ldots, a_2, a_1]$ and $[b_m, b_{m-1}, \ldots, b_2, b_1]$ are equal. The converse also holds. ●

Note: Since an arbitrary rational number corresponds to a trivial tangle, the trivial tangles are referred to as ***rational tangles***.

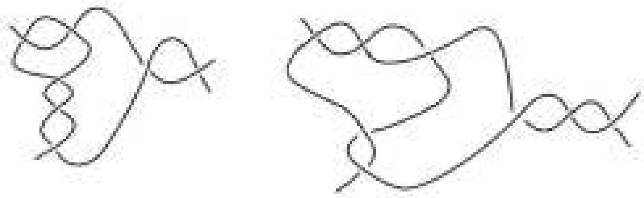
Example 3.2.3

Consider 2 tangles given by the sequence of integers -2 3 2 and 3 -2 3.

The continued fraction corresponding to -2 3 2 is $2 + \frac{1}{3+\frac{1}{-2}} = \frac{12}{5}$.

The continued fraction corresponding to 3 -2 3 is $3 + \frac{1}{-2+\frac{1}{3}} = \frac{12}{5}$.

Since the continued fractions are equal, these two rational tangles are equivalent.



-2 3 2   3 -2 3

Figure 3.24

Example 3.2.4

The tangle 3 2 -4 has continued fraction $-4 + \frac{1}{2+\frac{1}{3}} = \frac{-25}{7}$.

Thus this tangle is distinct from the 2 equivalent tangles -2 3 2 and 3 -2 3.

Example 3.2.5

Show that the rational tangle 2 1 a_1 $a_2 \ldots a_n$ is equivalent to the rational tangle -2 2 a_1 $a_2 \ldots a_n$.

The continued fractions are respectively,

$$a_n + \cfrac{1}{a_{n-1}+\cfrac{1}{\cdots\cdots + \cfrac{1}{a_1+\cfrac{1}{1+\frac{1}{2}}}}} = a_n + \cfrac{1}{a_{n-1}+\cfrac{1}{\cdots\cdots + \cfrac{1}{a_2+\cfrac{1}{a_1+\frac{2}{3}}}}}$$

and

$$a_n + \cfrac{1}{a_{n-1}+\cfrac{1}{\cdots\cdots + \cfrac{1}{a_1+\cfrac{1}{2+\frac{1}{-2}}}}} = a_n + \cfrac{1}{a_n-1+\cfrac{1}{\cdots\cdots + \cfrac{1}{a_2+\cfrac{1}{a_1+\frac{2}{3}}}}}$$

which are the same. Thus the two tangles are equivalent.

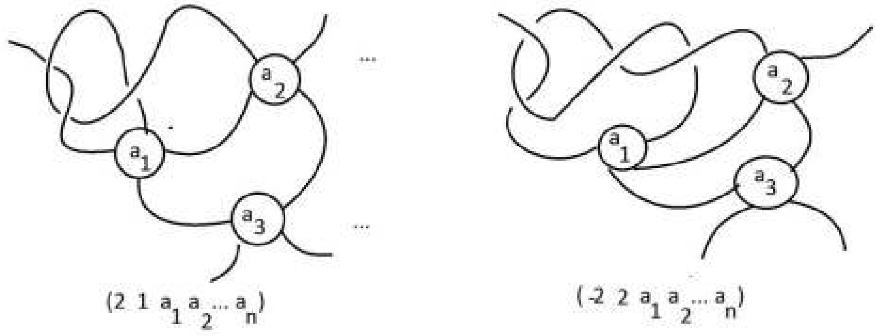


Figure 3.25

Reidemeister move III can be used to obtain the first tangle from the second.

If the ends of a rational tangle are closed off, then the resulting knot (or link) obtained is a ***rational knot (or link)***.

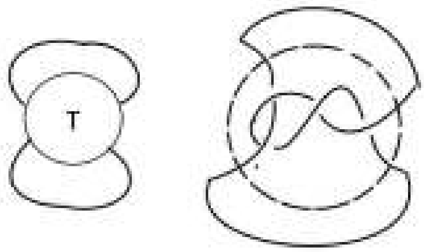


Figure 3.26 (rational knots)

For instance, the figure-eight knot is a rational knot obtained from the rational tangle 2 2. In fact, it is the numerator N(2 2).

The notation to denote rational tangles $a_1\, a_2 \dots a_n$ can be used to denote the corresponding rational knots. This notation is called the ***Conway's Notation***.

Example 3.2.6

The Conway's Notation for each of the knots in the figure 3.27 is as below.

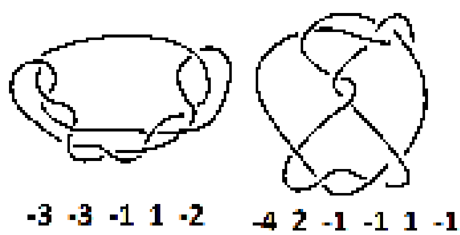


Figure 3.27

Binary operations on set of tangles

(1) Multiplying tangles

Reflect the first tangle across its NW to SE diagonal line and glue it to the second tangle.

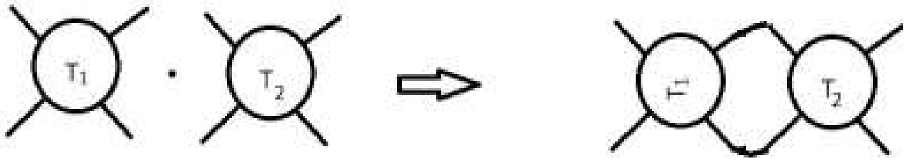


Figure 3.28

<u>Note</u>: Multiplying a rational tangle by an integer tangle will always generate a rational tangle.

A rational tangle (3 2) is obtained by multiplying together the two tangles 3 and 2.

If a tangle is reflected across its NW to SE diagonal line, then it is simply multiplying by the 0-tangle.

(2) Adding tangles

Two tangles can be added as in figure 3.29.

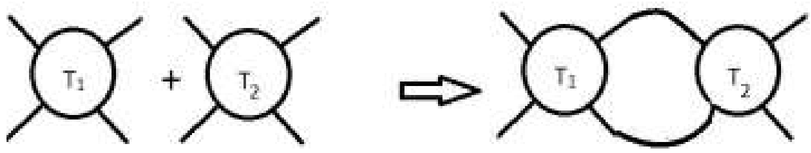


Figure 3.29

Example 3.2.7

The knot 8_5 can be written as the knot corresponding to the tangle   30+30+20, the sum of 3 rational tangles . If each tangle is multiplied by 0-tangle and then added together, the resultant tangle is denoted by the sequence of numbers that stand for the original tangles separated by commas. So the tangle 8_5 is denoted by 3, 3, 2.

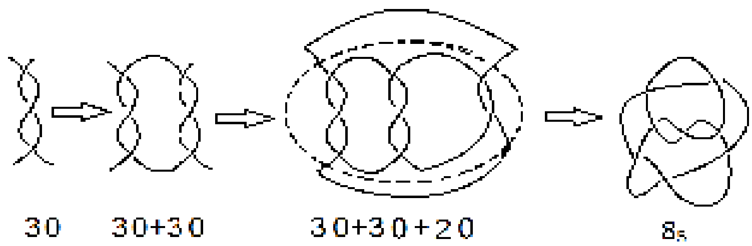


Figure 3.30

A knot obtained from a tangle reperesented by a finite number of integers separated by commas is called ***Pretzel Knot***.

Example 3.2.8

Draw the tangles 2, -3, 2, 4, 1 and -2, 3, 1, 4, 2 and the corresponding knots obtained connecting the NW string to the NE string and SW string to the SE ring.

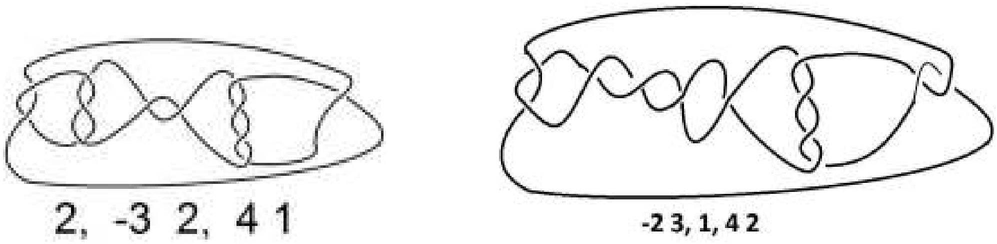


Figure 3.32

A tangle obtained by the operations of addition and multiplication on rational tangles is called an ***Algebraic Tangle***.

Example 3.2.9

Draw the algebraic tangle (3,2,1) . (1,2,2)

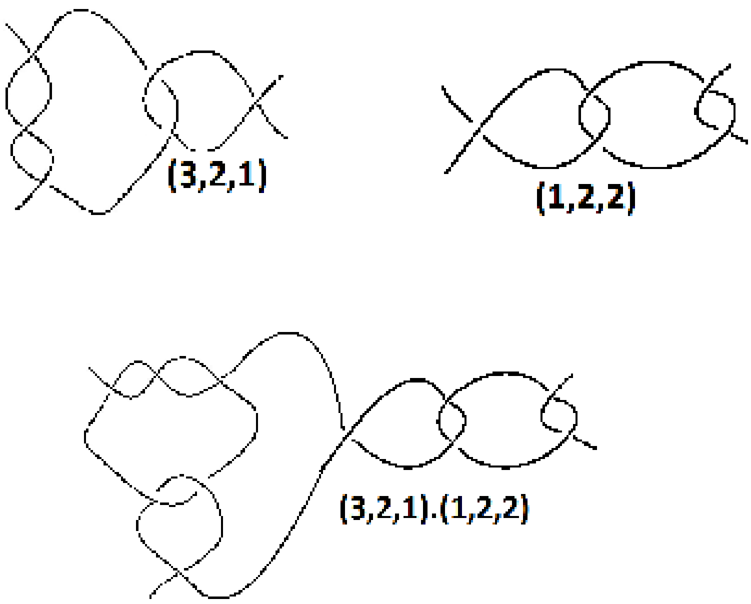


Figure 3.33

Algebraic Tangles behave like real numbers. They can be added or multiplied.

The 0-tangle is an additive identity for tangles.

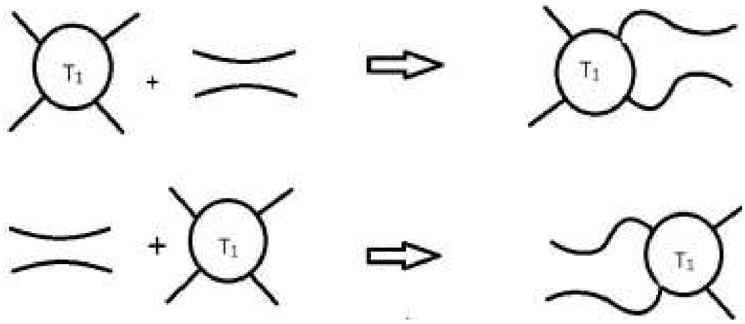


Figure 3.34

But in the case of the multiplicative identity, the left identity is the ∞-tangle whereas it is not a right identity.

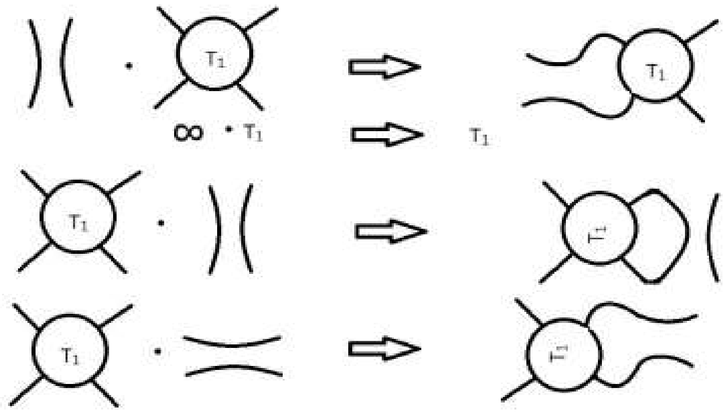


Figure 3.35

An ***algebraic link*** is the numerator of an algebraic tangle. Such a link is also called an ***Arborescent Link.***

<u>Note:</u> An algebraic knot with Conway Notation containing no negative signs must be an alternating knot.

The family of 2- bridge knots are often called rational knots due to the following theorem.

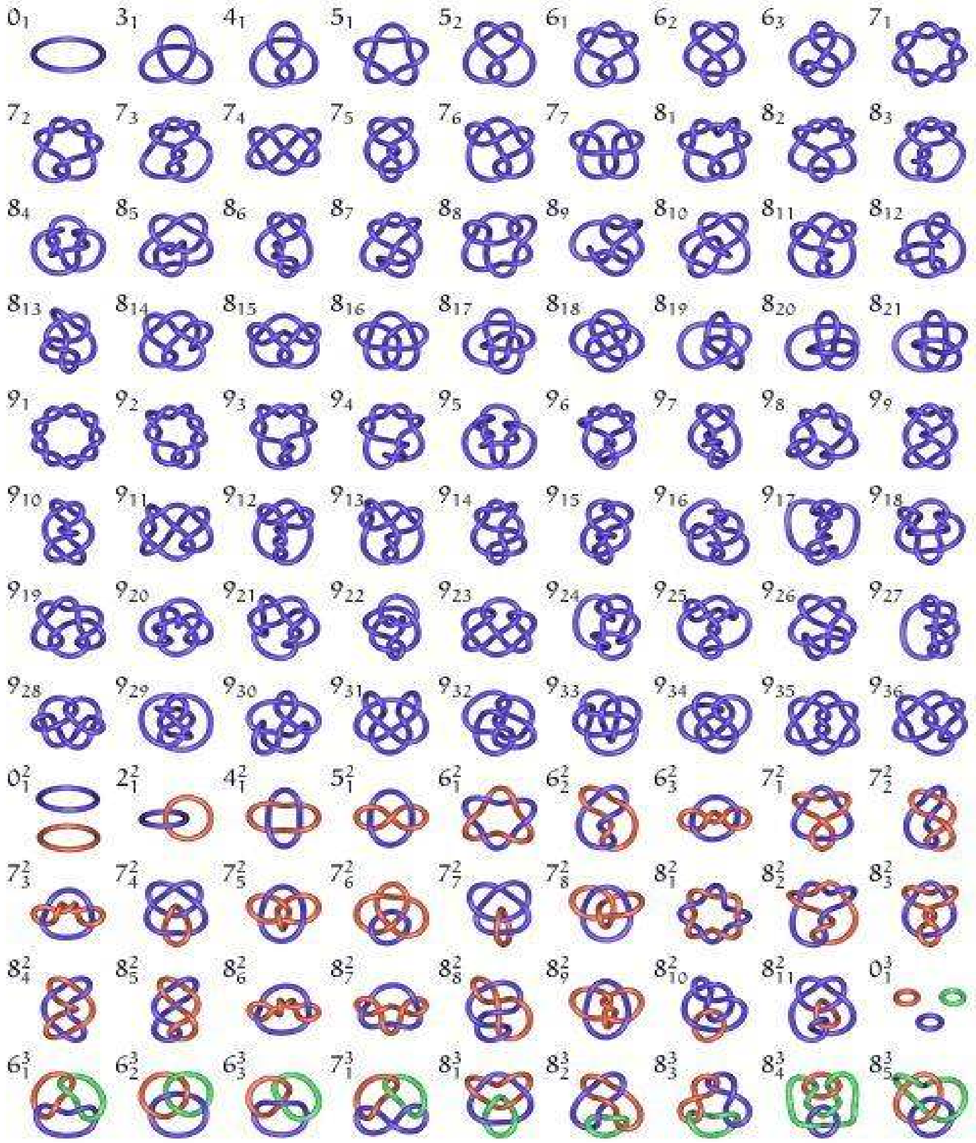
Theorem 3.2.2

(1) A 2- bridge knot (or link) is the denominator of some rational tangle.
(2) Conversely, the denominator of a rational tangle is a 2- bridge knot(or link). ●

As in the case of a rational tangle, a 2- bridge knot (or link) $a_1\ a_2 \ldots a_{2k+1}$ with $a_i \neq 0$ can be associated with a rational number $\frac{p}{q} = [a_1\ a_2 \ldots a_{2k+1}],\ g.c.d.(p,q) = 1.$

TABLE- 1

Table of projections of alternating knots up to 9 crossings and links up to 8 crossings.



Chapter- 4

SEIFERT SURFACES

In 1934, the German mathematician Herbert Seifert came up with an algorithm to create an orientable surface with one boundary component from a given knot such that the boundary circle is that knot. An invariant called the *genus of a knot* is defined using this surface which could be used to distinguish knots.

§. 4.1. Compact surfaces

A surface S is a two – manifold, that is, any object such that every point in that object has a neighbourhood in the object that is a disc, which is connected.

Figure 4.1 (the 2-sphere $S^2$and the torus)

The 2-dimensional unit sphere $S^2 = \{x \in \boldsymbol{R}^3 : |x| = 1\}$, a torus are all surfaces. If a surface is thought to be made of rubber then it can be deformed in shape. Such a rubber deformation is called an ***isotopy.*** If a surface can be obtained from another by a rubber deformation then the surfaces are said to be ***equivalent***. Two surfaces which are equivalent under an isotopy are called ***isotopic surfaces***.

Surfaces can be studied by cutting them into triangles. The triangles should fit together nicely on their edges so that they cover the entire surface (figure 4.2). Such a subdivision of a surface into triangles is called a ***triangulation***.

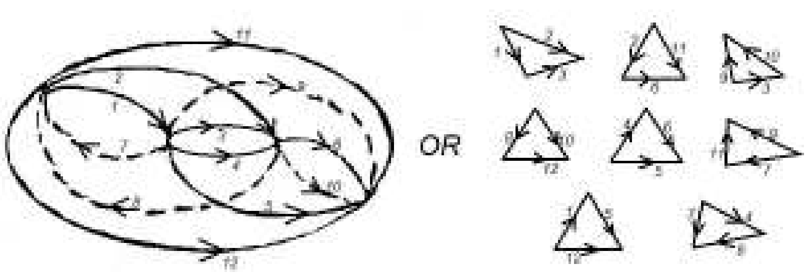


Figure 4.2 (triangulation of a torus)

Two surfaces are homeomorphic if one of them can be triangulated, then cut along a subset of edges into pieces, and glued together along the edges according to instruction given by the orientation and label of the edges in order to obtain the second surface.

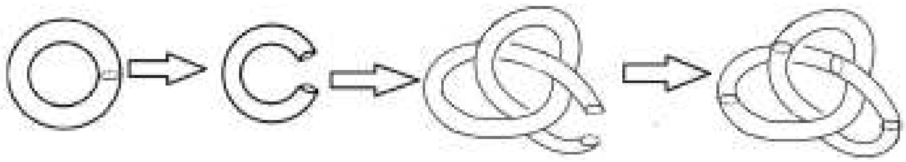


Figure 4.3 (sequences of figures show the homeomorphism between the two surfaces)

The 2– sphere S^2 and the torus T are not homeomorphic. To show this a numerical invariant called the *Euler characteristic* is used.

Definition 4.1.1

A surface S is said to be ***compact*** if it has a triangulation with finite number of triangles. The 2- sphere S^2, the torus are all compact surfaces.

Definition 4.1.2

Let S be a compact surface with triangulation T. Let V be the number of vertices in T, E the number of edges in T and F the number of faces (or triangles) in T. Then the ***Euler characteristic*** of the triangulation is $\chi = V - E + F$.

The figure 4.1.2 shows a triangulation of a torus. Here V= 4, E= 12, F= 8 so that $\chi = 0$.

Note: The Euler characteristic depends only on the surface and not on the particular triangulation of the surface.

Definition 4.1.3

Let S_1 and S_2 be disjoint surfaces. Their ***connected sum*** denoted by $S_1 \# S_2$, is formed by cutting a small circular hole in each surface and then gluing the two surfaces together along the boundaries of the holes. In other words, choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ where D_1 and D_2 are closed discs. Let S_i' be the complement of the interior of D_i in S_i, i= 1,2. Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2. Then $S_1 \# S_2$ is the quotient space of $S_1' \cup S_2'$ obtained by identifying the points *x* and *h(x)* for all *x* in the boundary of D_1.

Proposition 4.1.1

Let S_1 and S_2 be compact surfaces. The Euler cahracterisitc of S_1 and S_2 and their connected sum, $S_1 \# S_2$ are related by the formula

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_1) - 2. \qquad \bullet$$

Using the proposition, the connected sum of n tori has Euler characteristic 2-2n. This can be shown by induction on n.

The Euler characteristic of a sphere is 2.

Definition 4.1.4

The surface that is the connected sum of n tori is said to be of ***genus n*.**

A sphere is of genus 0. Any surface of genus n has Euler Charateristic of 2 – 2n.

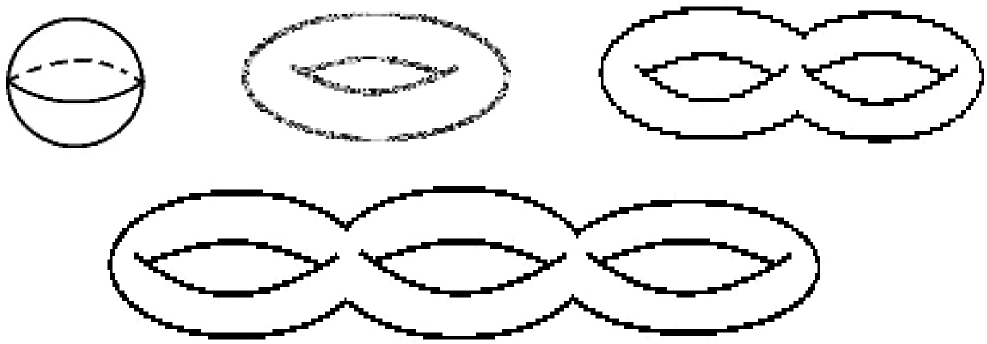


Figure 4.4 (a 2-sphere, a torus, connected sum of two tori and that of three tori)

§.4.2. Orientable and Non orientable Surfaces

Consider a rectangular strip as in the figure 4.5.

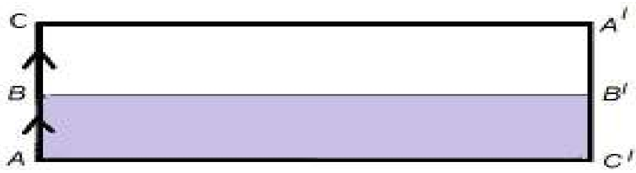


Figure 4.5

Glue ABC to A′B′C′ to construct a Mobius strip.

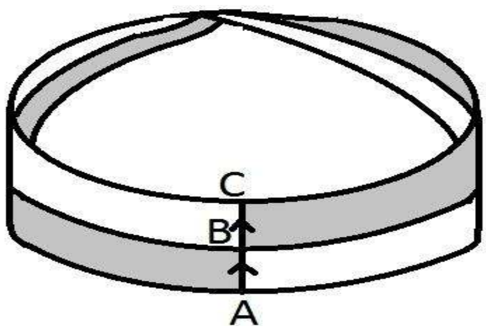


Figure 4.6 (a Mobius strip)

The central line of the strip becomes a circle after the gluing or identification of the two ends after making a half twist. If one starts at a point on this circle with a definite choice of orientation and goes round the circle once and comes back to the initial point, the original orientation would be reversed. Such a path in the manifold is called an ***orientation- reversing*** path. A closed path which does not have this property is called an ***orientation preserving*** path. Any closed path in a plane is orientation preserving.

A connected two-manifold is defined to be ***orientable*** if every closed path is orientation preserving.

A connected 2-manifold is ***non orientable*** if there exists at least one orientation-reversing path.

In other words, an orientable surface is 2-sided while a non orientable surface is 1-sided.

The connected sum of two orientable surfaces S_1 and S_2 is again orientable. But if either S_1 or S_2 is non orientable, then so is $S_1 \# S_2$.

Also, the Euler Characteristic of an orientable surface is always even, whereas that of a non- orientable surface is either even or odd.

The Euler characteristic is a topological invariant and the following result holds.

Theorem 4.2.1

Let S_1 and S_2 be compact surfaces. Then S_1 and S_2 are homeomorphic if and only if their Euler characteristics are equal and both are orientable or both are non orientable. ●

§. 4.3. Surfaces with Boundary

If the interiors of a finite number of disjoint closed discs in a compact surface are removed, then a ***bordered surface*** is obtained. The boundaries of the discs are left on the surfaces and these become the boundary of the surface. These boundaries are circles and are called ***boundary components***. In figure 4.7, the first surface has one boundary component and the second has two. Since deformation is allowed in topology, the shapes of the component can be different as in the figure 4.8.

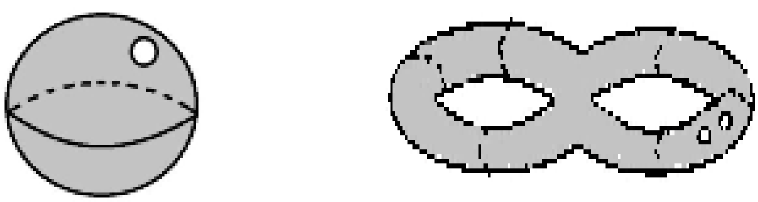


Figure 4.7 (bordered surfaces)

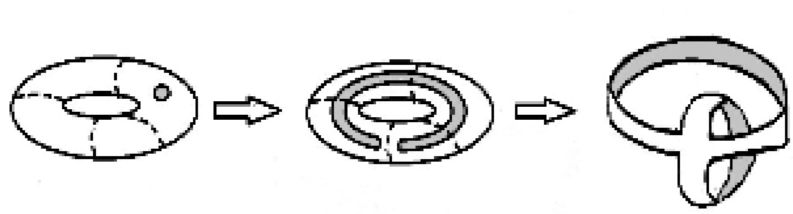


Figure 4.8 (surfaces with boundary components)

The topological type of a surface with boundary depends only on the number of boundary components and the topological type of the surface obtained by filling up the boundary components by gluing a disc on each boundary component or capping of the surface with the boundary. This gives the following theorem.

Theorem 4.3.1

Let M_1 and M_2 be the compact bordered surfaces, assume that their boundaries have the same number of components. Then M_1 and M_2 are hoemomorphic if and only if the surfaces M_1^* and M_2^* (obtained by gluing a disc to each boundary component) are homeomorphic. ●

The Euler Characteristic of a Bordered Surface

If M^* is a triangulated surface without boundary and the interiors of k disjoint triangles are removed to obtain a surface with boundary, M then

$$\chi(M) = \chi(M^*) - k.$$

Figure 4.9

The Euler Characteristic of the surface M_1 in figure 4.9 is

$\chi(M_1) = \chi(M_1^*) - 3 = 0 - 3 = -3$, where M_1^* is a torus of Euler Characteristic 0.

$\chi(M_2) = \chi(M_2^*) - 2 = -4 - 2 = -6$, where M_2^* is a surface of genus 3.

If the Euler characteristics of two bordered surfaces are different then the two surfaces are distinct. But unlike the surfaces without boundary surfaces, those with boundary cannot be distinguished purely Euler Characteristic. For instance, the two surfaces in the figure 4.10 are not homeomorphic but have the same Euler Characteristic, 1. The first surface is non-orientable but second one is orientable.

Figure 4.10 (different surfaces with same Euler Characteristic)

Thus the Euler Characteristic criterion alone is not sufficient to distinguish between surfaces with boundary leading to the following theorem.

Theorem 4.3.2

Two compact bordered surfaces are homeomorphic if and only if they have the same number of boundary components, they are both orientable or both non orientable, and they have the same Euler Characteristic. ●

Definition 4.3.1

The ***genus of a compact bordered surface*** M is defined to be the genus of the compact surface M^* obtained by attaching a disc to each boundary component of M.

The relation between the genus, g and the Euler Characteristic, χ of a compact surface is as follows:

$$g = \begin{cases} \frac{1}{2}(2-\chi) & if\ \ orientable \\ 2-\chi & if\ \ non\ orientable \end{cases}$$

If the surface is a compact bordered surface then the genus in terms of its Euler Characteristic, χ and the number of components of the boundary k is as follows:

$$g = \begin{cases} \frac{1}{2}(2-\chi-k) & if\ \ orientable \\ 2-\chi-k & if\ \ non\ orientable \end{cases}$$

§. 4.4. Seifert Surfaces

In knot theory Seifert introduced the genus of a knot as an invariant in order to classify knots. He used a connected oriented compact surface that has the knot as its boundary to define the genus of any kind or link. The following theorem ensures the existence of such a surface.

Theorem 4.4.1

Given an arbitrary oriented knot (or link) K, there exists in R^3 an orientable, connected surface that has as its boundary K.

Proof:

Seifert gave an algorithm to construct such a surface.

To construct an embedded surface in space for a particular knot K.

(i) Starting with a projection of the knot. Choose an orientation on K.

(ii) At each crossing of the projection, 2 strands come in and 2 strands go out. Eliminate the crossing by connecting each of the strands coming into the crossing to the adjacent strand leaving the crossing. The resultant strand will no longer cross.

(iii) The result will be a set of circles in a plane. These circles are called ***Seifert circles***. Fill in the circles to get discs in the plane. The discs are chosen at different heights rather than in the same plane so that they do not intersect each other.

(iv) Finally, connect the discs to one another at the crossings of the knot by twisted bands.

The result is a surface with one boundary component such that the boundary component is the knot. The surface is also orientable. Let the Seifert circle have orientation of the knot. For each disc that has a clockwise orientation on its bounding Seifert Circle paint its upward pointing face white and its downward pointing face black. For each disc that has counterclockwise orientation on its bounding Seifert Circle, paint its upward pointing face black and its downward pointing face white. At each crossing, the two discs are connected by a band with a half twist. If two discs are adjacent then they are of opposite orientation on their boundaries. If one disc is on top of the other then they are of the same orientation on their boundaries. Due to the half twist the entire surface can be painted black and white so that no black paint touches any white paint. Thus the surface is orientable.

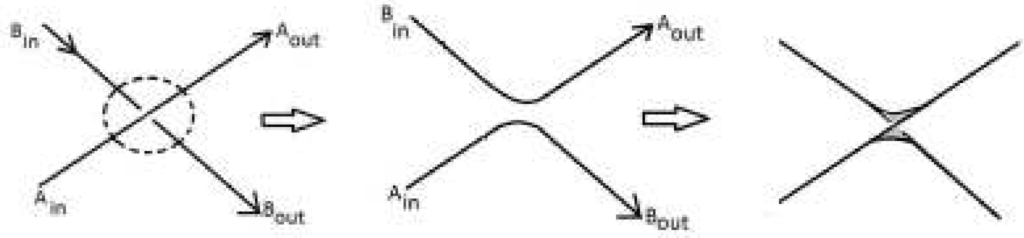


Figure 4.11 (construction of a Seifert surface)

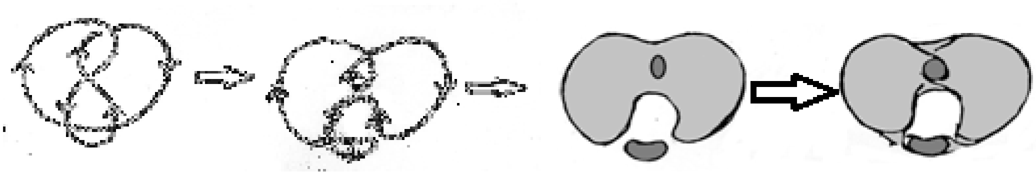


Figure 4.12 (Seifert surface of a figure-eight knot)

Definition 4.4.1

An orientable, connected surface that has as its boundary an orientable knot (or link) K is called a ***Seifert Surface*** of K.

Example 4.4.1

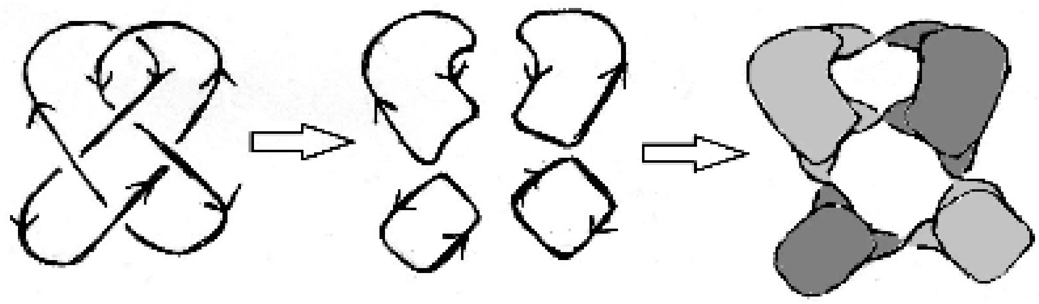


Figure 4.13 (Seifert surface of 5_2 knot)

Example 4.4.2

Figure 4.14 (Seifert surface of a 7-crossing knot)

Definition 4.4.2

The **genus** of a knot K is defined as the minimum genus of all Seifert surfaces for the given knot K. It is denoted by ***g(K)***.

The genus is a knot invariant. It is easier to find a genus of a Seifert Surface using the classical invariant, the Euler Characteristic χ. Since a Seifert Surface is an orientable surface with one boundary component, its genus

$$g = \begin{cases} \frac{1}{2}(2 - \chi - 1) = \frac{1}{2}(1 - \chi) \end{cases}$$

Theorem 4.4.2

If c is the number of crossings and s is the number of Seifert circles then $\chi = s - c$ and the genus of the surface is $g = (c - s + 1)/2$.

Proof:

Let the Seifert surface have no vertices. Add two vertices at each crossing, one on each strand. Join these vertices with edges. This gives c extra edges and two vertices. Apart from these edges there are two other edges from each vertex which gives 4c edges. But then, here each edge will be counted twice and so the number of other edges is only 2c. Thus there are $c + 2c = 3c$ edges, 2c vertices and s faces.

Therefore $\chi = V - E + F = 2c - 3c + s = s - c$

$$g = \frac{1}{2}(1 - \chi) = \frac{1}{2}(c - s + 1)$$

Thus the Seifert algorithm can be used to find the surfaces bounding the knots and the Euler characteristics can be used to identify the type of surfaces. This is illustrated in the following examples.

Example 4.4.1

Figure 4.15

The sequence of diagrams in figure 4.16 gives the Seifert Surface.

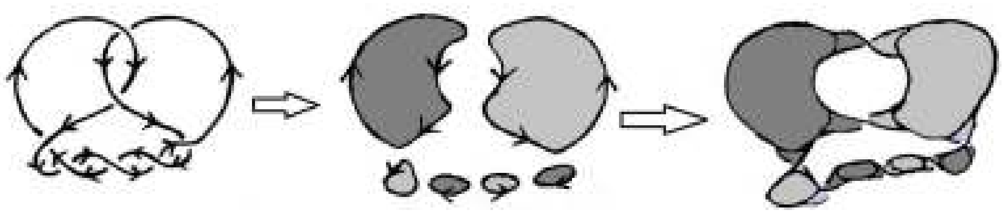


Figure 4.16

$\chi = s - c = 6 - 7 = -1$ so that $g = \frac{1}{2}(c - s + 1) = 1$

Thus the surface is of genus 1 which is a torus.

Example 4.4.2

Figure 4.17

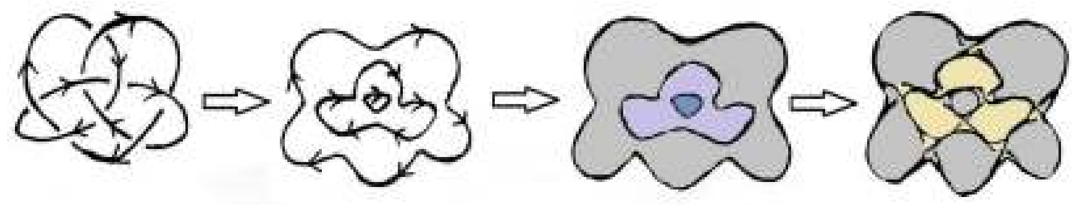


Figure 4.18

χ = $s - c = 6 - 7 = 3 - 8 = -5$ so that $g = \frac{1}{2}(c - s + 1) = 3$

Thus surface bounded by the knot is a surface of genus 3 which is 3-Tori.

Example 4.4.3

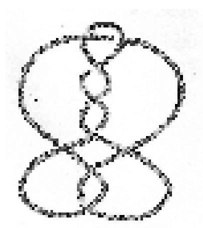


Figure 4.19

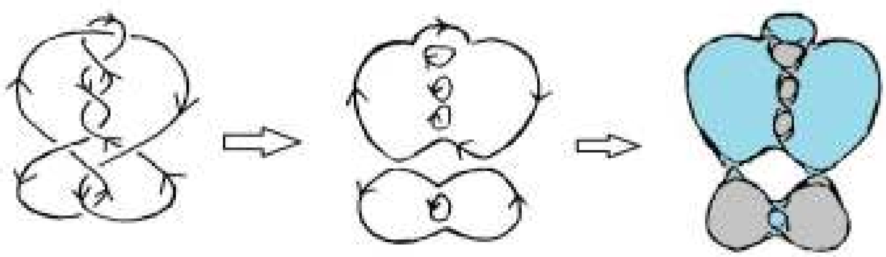


Figure 4.20

χ = $s - c = 6 - 9 = -3$ so that $g = \frac{1}{2}(c - s + 1) = 2$.

Thus surface bounded by the knot is a surface of genus 2 which is 2-Tori.

Example 4.4.4

The *Twist Knots* are the knots shown in the Figure 4.21.

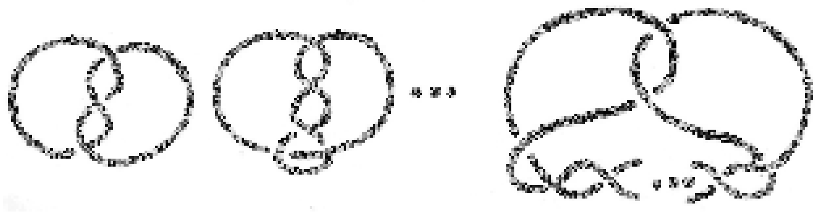


Figure 4.21

The first is a trefoil knot and the second a figure-8 knot. All of the Twist Knots have genus 1.

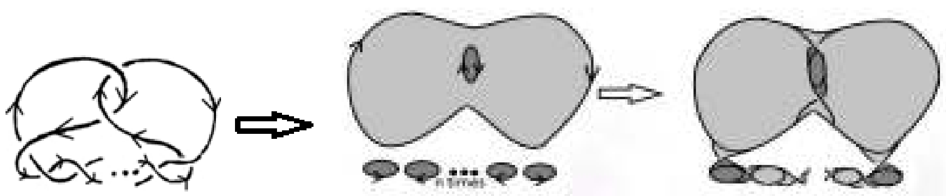


Figure 4.22

The number of Seifert Circles, s = n+2

Number of crossings, c = n+3

$\chi = s - c = -1$ so that $g = \frac{1}{2}(c - s + 1) = 1$

Theorem 4.4.3

Applying Seifert's Algorithm to an alternating projection of an alternating knot or link does yield a Seifert Surface of minimal genus. ●

The proof was given by David Gabai.

The genus of an unknot is 0. Thus it bounds a disc which can be capped off to form a sphere.

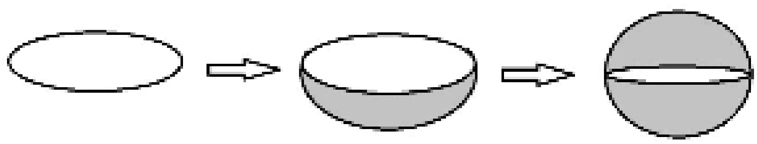


Figure 4.23

Theorem 4.4.4

Given 2 knots J & K,

$$g(J \# K) = g(J) + g(K).$$

Proof:

To show: $g(J \# K) \leq g(J) + g(K)$.

Let S_J be a Seifert Surface for J with minimal genus and S_K that for K. Now compose S_J and S_K along the boundary by removing a small piece along their boundaries and sewing them together. Then a Seifert Surface $S_{J\#K}$ is obtained whose boundary is $J \# K$.

Let S_J^* and S_K^* be the corresponding surfaces without boundary. Then $\chi(S_J^*) = 1 - 2g(J)$,

$\chi(S_K^*) = 1 - 2g(K)$.

Let T_1 be a triangulation of S_J with V_1 vertices, E_1 edges and F_1 faces and let T_2 be a triangulation of S_K with V_2 vertices, E_2 edges and F_2 faces. Then

$$\chi(S_{J\#K}) = (V_1 + V_2 - 2) - (E_1 + E_2 - 1) + F_1 + F_2$$

$$= \chi(S_J) + \chi(S_K) - 1$$

$$= 1 - 2g(J) + 1 - 2g(K) - 1$$

$$= 1 - 2(g(J) + g(K))$$

Let $S_{J\#K}^*$ be the corresponding surface without boundary. Then

$$\chi(S_{J\#K}^*) = \chi(S_{J\#K}) + 1$$

$$= 2 - 2(g(J) + g(K))$$

$$\Rightarrow \quad g(S_{J\#K}) = \frac{1}{2}[1 - \chi(S_{J\#K}^*)]$$

$$= \frac{1}{2}[-1 + 2(g(J) + g(K))]$$

$$\leq g(J) + g(K)$$

$$\Rightarrow \quad g(J\#K) \leq g(J) + g(K) \ .$$

To prove: $g(J\#K) \geq g(J) + g(K)$

Let S be a Seifert Surface of minimal genus for J#K. Since J#K is a composite knot there is a sphere F which separates J and K such that one, say, J lies inside Fand K outside and F intersects J#K only at two points. The sphere F will intersect the Seifert Surface S.

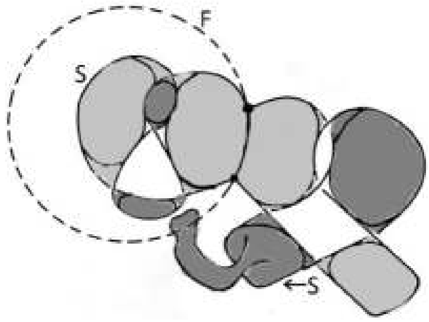


Figure 4.24

Along with the two points of intersections of F and J#K, an arc joining the two points also lie on F$\cap S$. The intersection of F and S can also be points or loops. Deform S so that it does not touch F at a point. Now the set F$\cap S$ contains only arcs and loops. This small deformation of S so that F$\cap S$ contains only arcs and loops is called putting the surface in general position. Since F intersects J#K only at two points there will be only one arc of intersection between F and S.

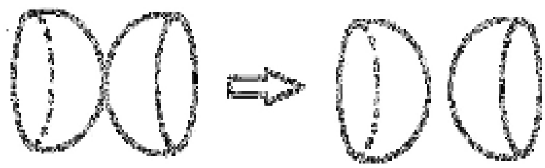


Figure 4.25 (A single point intersection can be removed)

Every intersection loop on F bounds a disc on F. Let C be the intersection loop that is innermost on F, that is, C bounds a disc on F containing no other intersecting curves.

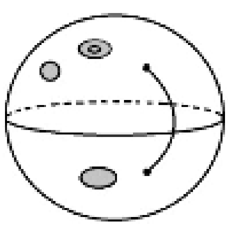


Figure 4.26

Cut open S along C to obtain two copies of C in the cut open S. Glue a disc to each copy of C. Two new surfaces, say, $S \, and \, \hat{S}$ are obtained which may or may not be connected. One of them, say $\hat{S}$ will be a Seifert Surface for J#K. This surface will intersect F in lesser number of circles. If S' is not connected to the knot then it can be thrown off. Also,

$$g(S) = g(S') + g(\hat{S})$$

Therefore $g(S') = 0$ since if not, then $g(S) \geq g(S')$ which is a contradiction to the hypothesis that $g(S)$ is minimal. Thus $g(S) = g(S')$

Repeat this surgery operation till $S \cap F$ has no intersection circles. Thus $S \cap F$ contains only one arc of intersection. Hence F divides S into two Seifert Surfaces $S_J \, and \, S_K$ respectively for J and K and

$$g(S_J) + g(S_K) = g(S) = g(J\#K)$$

Since $g(S_J) \geq g(J) and \, g(S_K) \geq g(K)$

$$g(J\#K) \geq g(J) + g(K)$$

Hence, $g(J\#K) = g(J) + g(K).$ ●

<u>Note:</u>

(1) A trivial knot cannot be the composition of two non trivial knots.

For, suppose that O is an unknot, then g (O) = 0 and g(K)>0 if K≠O

Suppose that O= J#K where J, K≠O,

Then by the theorem 4.4.4,

$$g(O) = g(J) + g(K) > 0, \quad \text{a contradiction.}$$

(2) The knots of genus 1 are prime.

For, let K be a knot such that $g(K) = 1$. If K is not prime, then K is the composition of two knots, say, $K_1 \, and \, K_2$, that is, $K \approx K_1 \# K_2$.

Then $1 = g(K) = g(K_1) + g(K_2)$

Therefore either $g(K_1) = 1 \, and \, g(K_2) = 0 \, or \, g(K_1) = 0 \, and \, g(K_2) = 1$

In either case one of them is an unknot. Hence K is prime.

The Seirfeit’s Algorithm can be used to find a minimal genus Seifert Surface for an alternating knot. But there are other types of knots too. An Israeli mathematician, Yoan Moriah proved that there are knots for which minimal genus Seifert Surface cannot be obtained by applying Seifert Algorithm to any projection of the knot.

David Gabai from Caltech showed that the pair of knots called the Kinoshita Terasaka mutants do not have the same genus and hence must be distinct knots.

Chapter-5

POLYNOMIALS

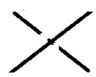
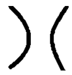
§. 5.1. The Bracket and Jones polynomial

A way to distinguish knots is by associating a polynomial to each knot. A polynomial can be computed from a knot diagram. Any two different knot diagram of the same knot will give the same polynomial and so the polynomial is an invariant of the knot. Here the polynomial is a Laurent Polynomial which has both positive and negative powers of *t*.

The first polynomial associated to knots and links was due to J. Alexander in 1928. John Conway found a method to calculate the Alexander Polynomial using a relation called ***Skein Relation***. This is an equation that relates the polynomial of a link to polynomial of links obtained by changing the crossing in the projection of the original link.

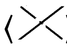
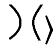
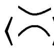
In 1984, Vaughan Jones discovered another polynomial for knots and links. This was followed by the discovery of a polynomial called the HOMFLY polynomial named after its discoverers Hoste, Ocneau, Millet, Freyd, Lickorish and Yetter.

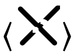
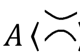
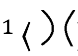
Louis Kauffman developed an easy way to construct Jones Polynomial by introducing a polynomial associated to knots called *bracket polynomial*.

A ***bracket polynomial*** of a knot K is denoted by < K >. The bracket polynomial of a link is obtained from the bracket polynomials of simpler link using a skein relation. If a crossing of a link projection is given it can be split open vertically and horizontally so that the two new link projections will have one less number of crossings. If the crossing is of the form  then it is split as  and .

The bracket polynomial of a knot or a link is a Laurent Polynomial in one variable A which is determined by three rules:

Rule 1: $\langle \bigcirc \rangle = 1$

Rule 2: $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

Rule3: $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$

The first rule states that the polynomial of a trivial knot is 1.

The second rule resolves the crossings. Applying this rule recursively, the bracket polynomial can be computed by reducing the number of crossings eventually reaching a diagram with no crossings.

The third rule states that if a trivial component is added to a link without entangling it to the link then the polynomial of the resulting link is obtained by multiplying the polynomial of the original link by$(-A^2 - A^{-2})$.

<u>Note</u>: Rule 1 holds only for the knot diagram with no crossing.

Example 5.1.1

The bracket polynomial of the usual projection of the trivial link with n components is$(-1)^{n-1}(A^2 + A^{-2})^{n-1}$.

<u>Solution</u>

This can be shown by mathematical induction

By the Rule 3 if L is a unknot

$$\langle \bigcirc \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle \bigcirc \rangle$$

$$= (-A^2 - A^{-2}) \qquad \text{(by rule 1)}$$

If L is the trivial link of 2 components then by Rule 3

$$\langle \bigcirc \cup \bigcirc \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle \bigcirc \cup \bigcirc \rangle$$

$$= (-1)^2(A^2 + A^{-2})^2$$

Let $\langle \underbrace{\bigcirc \cup \bigcirc \cup \ldots \cup \bigcirc}_{k\ times} \rangle = (-1)^{k-1}(A^2 + A^{-2})^{k-1}$

Now let L be the trivial link with k components. Then

$$\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$$

$$= (-1)^k(A^2 + A^{-2})^k$$

which is the polynomial of the trivial link with (k+1) components. Thus for any trivial link with n components the Bracket polynomial is $(-1)^{n-1}(A^2 + A^{-2})^{n-1}$.

Example 5.1.2

Compute the Bracket Polynomial of the nontrivial link with 2 components, the Hopf Link.

$$\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$$

$$= A\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right) + A^{-1}\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right)$$

$$= A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2}(-A^2 - A^{-2})$$

$$= (-A^4 - A^{-4})$$

Example 5.1.3

Find the Bracket Polynomial of the trefoil knot.

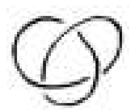


Figure 5.1

$$\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$$

$$= A\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right) + A^{-1}\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right)$$

$$= A^2\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right) + (A\langle\ \rangle + A^{-1}\langle\ \rangle) + \left(A\langle\ \rangle + \langle\ \rangle\right) + A^{-2}\left(A\langle\ \rangle + A^{-1}\langle\ \rangle\right)$$

$$= A^3(A^2 + A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2}) + A^{-1}.1 + A(-A^2 - A^{-2}) + A^{-1}.1 + A^{-1} + A^{-3}(-A^2 - A^{-2})$$

$$= A^3(A^4 + 2 + A^{-4}) - A^3 - A^{-1} + A^{-1} - A^3 - A^{-1} + A^{-1} + A^{-1} - A^3 - A^{-1} - A^{-5}$$

$$= A^7 - A^3 - A^{-5}$$

Lemma 5.1.1

If the link diagrams D and D' are related by one application of a Type II Reidemeister Move, then $\langle D\rangle = \langle D'\rangle$.

Proof

$\langle \; \rangle = A \langle \; \rangle + A^{-1} \langle \; \rangle$

$= A \left(A \langle \; \rangle + A^{-1} \langle \; \rangle \right) + A^{-1} \left(A \langle \; \rangle + A^{-1} \langle \; \rangle \right)$

$= A^2 \langle \; \rangle + (-A^2 - A^{-2}) \langle \; \rangle + \langle \; \rangle \langle \; \rangle + A^{-2} \langle \; \rangle = \langle \; \rangle \langle \; \rangle$

Thus the bracket polynomial is unchanged under a Type II move.

Lemma 5.1.2

If the link diagrams D and D' are related by one application of Type III Reidemeister Move, then $\langle D \rangle = \langle \mathrm{D}' \rangle$.

Proof

$\langle \; \rangle = A \langle \; \rangle + A^{-1} \langle \; \rangle$

$\langle \; \rangle = A \langle \; \rangle + A^{-1} \langle \; \rangle$

Since the bracket polynomial is invariant under Type II Reidemeister move $\langle \; \rangle = \langle \; \rangle$. Thus the bracket polynomial is invariant under Type III Reidemeister move.

The Bracket Polynomial is invariant under type II and III Reidemeister moves but it is not invariant under Type I move. For, consider

$\langle \; \rangle = A \langle \; \rangle + A^{-1} \langle \; \rangle = A(-A^2 - A^{-2}) \langle \; \rangle + A^{-1} \langle \; \rangle = -A^3 \langle \; \rangle$

$\langle \; \rangle = A \langle \; \rangle + A^{-1} \langle \; \rangle = A \langle \; \rangle + A^{-1}(-A^2 - A^{-2}) \langle \; \rangle = -A^{-3} \langle \; \rangle$

In order to resolve this issue a number called ***writhe*** is associated to the link diagram. The knot or link projection is given an orientation. Each crossing of the first type in the figure 5.2 is counted as +1 and the second type as -1.

(a) (b)

Figure 5.2

The sum of the +1s and -1s is called the ***writhe*** of the oriented link diagram L (or twist of the projection) and is denoted by ***w(L)***.

Example 5.1.4

The writhe of the link in the figure 5.3 is $w(L)= 3\text{-}5= -2$.

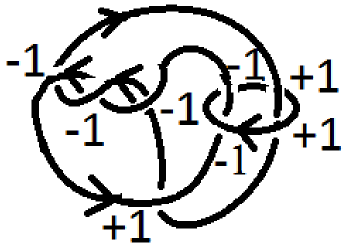


Figure 5.3

The writhe of a link projection is invariant under Reidemeister moves II and III. For, consider the move II and III in the figure 5.4.

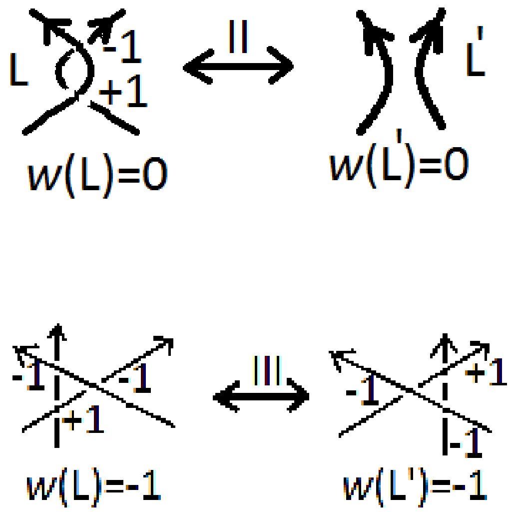


Figure5. 4

The Reidemeister move I always changes the writhe by± 1 as in the figure 5.5.

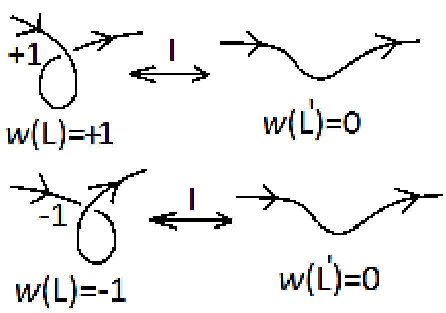


Figure 5.5

Definition5.1.1

A polynomial called ***X polynomial*** of oriented links is defined to be

$X(L) = (-A^3)^{-w(L)}\langle L\rangle$.

w(L) and X(L) are unaffected by the moves II and III. Therefore, X(L) is also unaffected by these moves.

X(L) is unaffected by move I as well. For, suppose that a link L' has a strand as in the figure 5.6 and L is a link obtained by removing the twist by move I. Then

w(L')=w(L)+1 so that

$X(L') = (-A^3)^{-w(L')}\langle L'\rangle$

$= (-A^3)^{-(w(L)+1)}\langle L'\rangle$

$= (-A^3)^{-(w(L)+1)}(-A^3)\langle L\rangle$ by move I on Bracket Polynomial.

$= (-A^3)^{-w(L)}\langle L\rangle$

= X(L)

Therefore, X(L) is an invariant for knots and links.

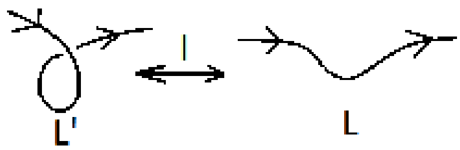


Figure 5.6

$X(\bigcirc\bigcirc) = (-A^3)^{-w(\bigcirc\bigcirc)}\langle \bigcirc\bigcirc\rangle$

$= (-A^2 - A^{-2})$

since the writhe of the trivial link with 2 components is 0.

Example 5.1.5

Compute X(L) of the oriented links in the figure 5.7.

Figure 5.7

If L is trefoil knot then *w(L)* = -3 so that
$X(L) = (-A^3)^3\langle L\rangle = -A^9(A^7 - A^3 - A^{-5}) = -A^{16} + A^{12} + A^4$

If L is Hopf Link where *w(L)* = 2.

$X(L) = (-A^3)^{-2}(-A^4 - A^{-4}) = (-A^{-2} - A^{-10})$

Definition 5.1.2

The ***Jones Polynomial*** is an X polynomial obtained by replacing A by $t^{\frac{-1}{4}}$. The Jones Polynomial of an oriented link L is denoted by $\boldsymbol{V_L(t)}$**.**

Example 5.1.6

From the knot diagram of the trefoil knot, the Jones polynomial of the trefoil knot is

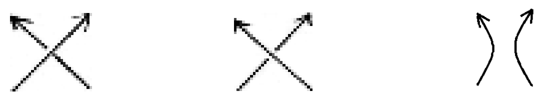
$$V_L(t) = -(t^{\frac{-1}{4}})^{16} + (t^{\frac{-1}{4}})^{12} + (t^{\frac{-1}{4}})^4$$

$$= -t^{-4} + t^{-3} + t^{-1}$$

The Jones Polynomial of the Hopf Link is
$$V_L(t) = -(t^{\frac{-1}{4}})^{-2} + (t^{\frac{-1}{4}})^{-10}$$

$$= -t^{\frac{1}{2}} + t^{\frac{5}{2}}$$

Let $L_+,\ L_-,\ L_0$ be three oriented link projections that are identical except at one of the crossing where they appear as in figure 5.8.



L_+ $\quad L_-$ $\quad L_0$

Figure5. 8

The Jones Polynomial satisfy the skein relation

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) + \left(t^{\frac{-1}{2}} - t^{\frac{1}{2}}\right)V_{L_0}(t) = 0.$$

This can be shown using the skein relation of the bracket polynomial.

$$\langle L_+\rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$$

$$\langle L_-\rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$$

$$X(L_+) = (-A^3)^{-w(L_+)}\langle L_+\rangle$$

$$= (-A^3)^{-1}\big(A\langle\,)(\, \rangle + A^{-1}\langle \asymp \rangle\big)$$

$$=-A^{-2}\langle\,)(\, \rangle - A^{-4}\langle \asymp \rangle$$

$$X(L_-) = (-A^3)^{-w(L_-)}\langle L_- \rangle$$

$$= (-A^3)^{+1}\big(A\langle \asymp \rangle + A^{-1}\langle\,)(\, \rangle\big)$$

$$=-A^4\langle \asymp \rangle - A^2\langle\,)(\, \rangle$$

$$A^4X(L_+) - A^{-4}X(L_-) = -A^2\langle\,)(\, \rangle + A^{-2}\langle\,)(\, \rangle$$

$$= (-A^2 + A^{-2})\langle\,)(\, \rangle$$

$$A^4X(L_+) - A^{-4}X(L_-) = (-A^2 + A^{-2})X(L_0)$$

Replacing A by $t^{\frac{-1}{4}}$ the equation in terms of Jones polynomial is obtained as

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) + \left(t^{\frac{-1}{2}} - t^{\frac{1}{2}}\right)V_{L_0}(t) = 0$$

There are certain characteristics of Jones Polynomial.

1. Suppose that $K_1\#K_2$ is the connected sum of two knots (or links).Then $V_{K_1\#K_2}(t) = V_{K_1}(t)V_{K_2}(t)$.
2. Suppose –K is the knot with the reverse orientation to that on K. Then $V_{-K}(t) = V_K(t)$.
3. Suppose $K^*$is the mirror image of a knot (or link) K. Then $V_{K^*}(t) = V_K(t^{-1})$

§. 5.2. The Alexander Polynomial

The first polynomial for knots was the Alexander polynomial invented in 1928. In 1969, John Conway showed a method to compute the Alexander polynomial Δ by the two rules.

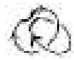
Rule 1: $\Delta(\bigcirc) = 1$

Rule2: $\Delta(L_{+1}) - \Delta\left(L_{-1}\right) + \left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)\Delta\left(L_0\right) = 0$

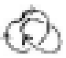
Here Rule 1 holds for any projection of the trivial knot.

Example 5.2.1

Compute the Alexander polynomial of the trefoil knot.

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) + \left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)\Delta($  $) = 0$

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = 1$

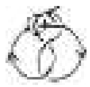
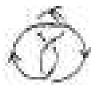
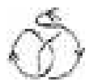
$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) + \left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)\Delta($  $) = 0$

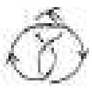
Now $\Delta\Big($  $\Big) = 0$ so that $\Delta\Big($  $\Big) = -t^{\frac{1}{2}} + t^{\frac{-1}{2}}$

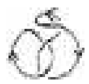
Therefore $\Delta\Big($  $\Big) = 1 + (t^{\frac{1}{2}} - t^{\frac{-1}{2}})^2 = t - 1 + t^{-1}$

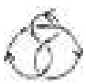
Example 5.2.2

Compute the Alexander Polynomial of figure-eight knot.

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) + \left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)\Delta($  $) = 0$

$\Delta\Big($  $\Big) = 1$

$\Delta\Big($  $\Big) = \Delta\Big($  $\Big) = t^{\frac{1}{2}} - t^{\frac{-1}{2}}$.

Thus $\Delta\Big($  $\Big) = -\left(t^{\frac{1}{2}} - t^{\frac{-1}{2}}\right)^2 + 1$

$= 3 - t - t^{-1}$.

Example 5.2.3

The Alexander polynomial of a splittable link is 0.

Remark:

Unlike the Jones polynomial there are nontrivial knots with Alexander polynomial equal to 1. For instance, (-1, 5, 7) – Pretzel Knot has Alexander polynomial 1. Thus it cannot distinguish all knots from the trivial knot. This is a limitation the Alexander polynomial.

Many polynomials have been discovered to distinguish knots (or links) from each other.

Chapter- 6

APPLICATIONS OF KNOT THEORY

§. 6.1. Knots in DNA

A major breakthrough in genetics happened in 1950s when James Watson and Francis Crick discovered the double helix structure of DNA. This structure helped to reveal the DNA replication. Deoxyribonucleic acid (DNA) molecule consists of a pair of long linear molecular strands intertwined along a linear axis forming a double helix. The molecular strand consists of alternate sugar and phosphate molecules. The nucleic acid is made up of a chain of linked units called nucleotides. Each nucleotide consists of the sugar, deoxyribose, made up of five carbon atoms. This sugar ring forms bonds to a single phosphate group between the third and the fifth carbon atom of adjacent sugar ring. The four bases found in DNA are Adenine (A), Thymine (T), Cytosine (C) and Guanine (G). The shapes and the chemical structure of these bases allow hydrogen bonds to be formed efficiently between A and T and between C and G. These bonds along with base stacking interactions hold the DNA strand together. Each base is attached to the first carbon atom in the sugar ring to complete the nucleotide.

The bonds between the sugars and phosphate group give a direction to the DNA strand. The asymmetric ends of the strands are called 5' and 3' ends with 5' end having a phosphate group attached to the 5^{th} carbon atom of the sugar ring and 3' end with the terminal hydroxyl group attached to the 3^{rd} carbon atom of the sugar ring. The direction of the DNA strand is read from 5' to 3'. In a double helix the direction of one strand is opposite to the direction of the other strand, that is, the strands are anti-parallel.

Besides the standard linear form, the molecule of DNA can take the form of a ring known as the circular DNA. The geometry of a cyclic duplex DNA can be modelled as a ribbon in 3-space with the 2 ends of the ribbon glued together. The two boundaries of the ribbon correspond to the 2 strands. Mathematically the model of the circular DNA is an annulus, R, an object that is topologically equivalent to $S^1 \times [-1,1]$. The axis of R is $C = S^1 \times \{0\}$. An orientation can be chosen for the axis of R and ∂R is considered to have the same orientation.

If the two ends of the linear duplex DNA are brought together to form a cyclic duplex DNA the 3' site must be glued to the 5' site and vice versa. So each strand of DNA glues its head to its own tail rather than to the tail of the other strand. Thus there must be only an even number of half twists in a circular DNA and it cannot be a Mobius strip.

The number of twists of a ribbon R along its axis is called the ***twisting number*** denoted by **Tw(R)**.

If the axis of R is viewed as spatial curve, then the sum of the ± 1s occurring at the crossings were the axis crosses itself is defined to be the ***signed crossover number*** for any particular projection.

The ***writhe*** of the ribbon R is measured as the average value of the signed crossover number, over every possible projection of the axis. It is denoted by **Wr(R).**

If the two boundaries of the ribbon are treated as the components of a link then the ***linking number*** denoted by **Lk (R)**, of the oriented link formed by the two components with parallel orientation is one half of the sum of the ± 1s occurring at the crossings between the two components.

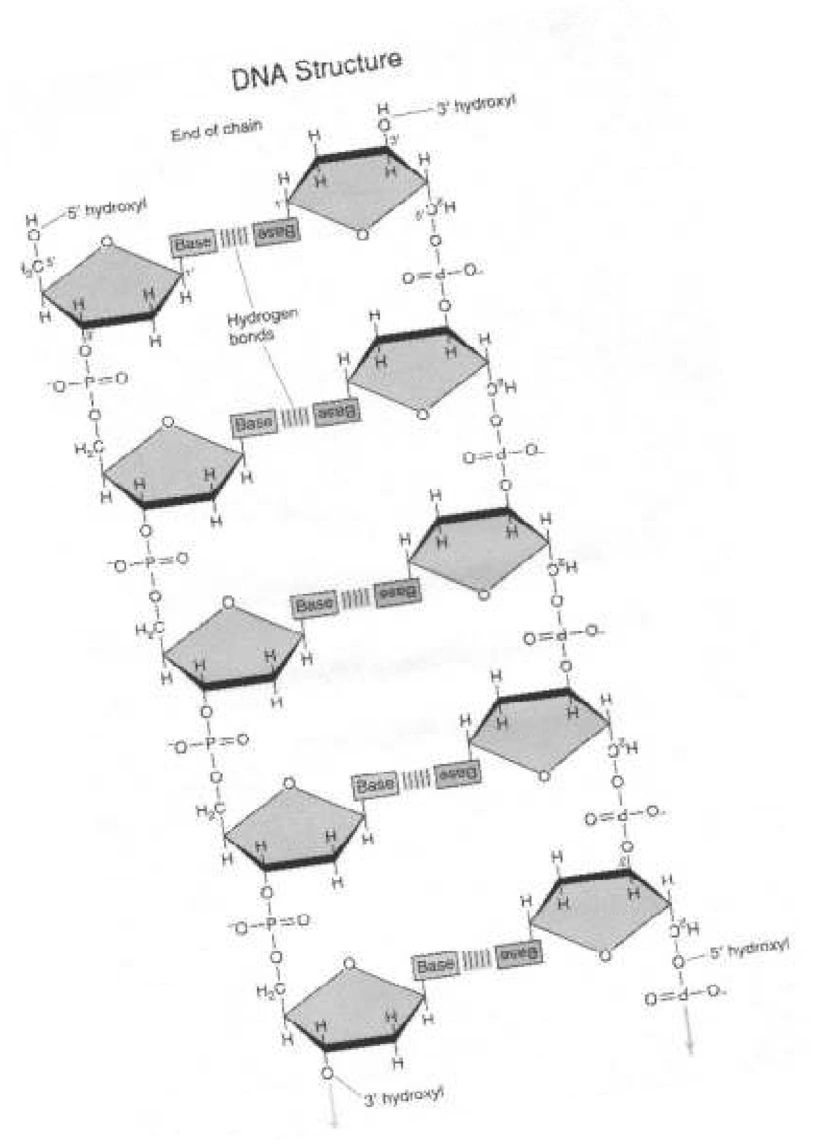


Figure 6.1 (The two strands of DNA read from 5' end to 3' end)

Figure 6.2 (A ribbon model of the cyclic duplex DNA)

The linking number which is an invariant does not depend on the particular placement of the link in space. The relation between these 3 invariants is given by

$$Lk(R) = Tw(R) + Wr(R) \qquad \ldots\ldots \qquad (1)$$

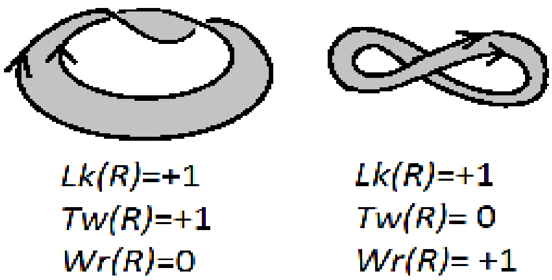


Figure 6.3

DNA twists around its axis at a rate of 10.5 base pair per helical twist in its relaxed state. This relaxed rate of twisting is caused by the way the sugars, phosphates and base pairs bond. Thus a cyclic duplex with 105 base pairs could lie flat in the plane as a 2-braid link with $Tw(R)$ =10, $Wr(R) = 0, Lk(R) = 10$. Sometimes a cyclic duplex DNA is more tightly twisted than 10.5 base pairs per twist. Suppose a cyclic duplex DNA is such that $Wr(R) = 0$ but both the twists and linking numbers are doubled so that $Tw(R) = Lk(R) = 20$ while the number of base pairs remain the same at 105. Since the number of base pairs is fixed the DNA would have to reduce the twist $Tw(R)$ to 10 to bring it to a relaxed state. In the process $Wr(R)$ will have to be increased to 10 using equation (1). Thus the axis of the DNA ribbon is contorted in space. This effect is called ***super coiling***. A DNA molecule is ***super coiled*** when $Wr(R) \neq 0$. A circular DNA appears negatively supercoiled under an electron microscope.

A nick is a discontinuity in one strand of a double stranded DNA molecule. This discontinuity is a missing bond between adjacent nucleotides of the same strand. A nick can be the result of damage or enzyme action. If a circular DNA is nicked then there is no linking number since at least one of the strands will not be a closed curve. This relaxes the DNA since it can choose its preferred twist and writhe.

DNA is kept very compactly in the nucleus. A cyclic DNA has three main topological forms, namely, super coiled, knotted, catenated or a combination of these. Replication, transcription and recombination are some biological functions performed by DNA. Replication is a process of reproducing a given DNA molecule. During transcription a copy of sequence of DNA is created. Recombination involves modifying DNA molecules. The DNA molecule becomes entangled at the end of the replication process. An enzyme called *Topoisomerases* helps in the packing of DNA in the nucleus and in the unknotting and unlinking of DNA links. Isomers are two molecules with same chemical combination but different structure. So two DNA molecules with the same base pairs but with different linking number are isomers but they are not topologically equivalent. So these DNA molecules are called *topoisomers*. The enzyme that causes the linking number to change is called *topoisomerase*. The enzyme cuts the DNA at one place, then a segment of DNA passes through this cut and finally the DNA reconnects itself. These do not change the chemical composition and connectivity of DNA, but changes its topology.

Recombination is a process in which genetic exchange of DNA takes place when an enzyme called *recombinase* rearranges the DNA sequences. In site-specific recombination the enzyme attaches to two specific sites on two strands of DNA called recombination sites. It then draws the sites together cuts open the two strands and recombines the four ends in some manner.

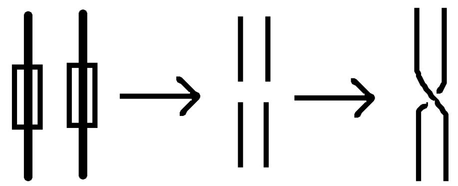


Figure 6.4 (site- specific recombination)

The circular DNA molecule before the action of the enzyme is called the ***substrate*** and that after the recombination is called the ***product***. The process of going from the cyclic DNA molecule to the state when the sites of the DNA are drawn together is called ***writhing process***. During this process the enzyme combines with the substrate. The resulting combined complex is called the ***synaptic complex***. Within this complex an orientation is assigned to the two strands using the DNA sequence of the recombination site. If the two sites are oriented in the same direction then the sites are called ***direct repeats***.

Action of enzymes on direct repeats usually results in a change in number of components so that knots are taken to links and links to knots or links of higher number of components.

If the two sites are oriented in opposite direction the sites are called ***inverted repeats***. Action of enzymes on inverted repeats usually results in no change in number of components.

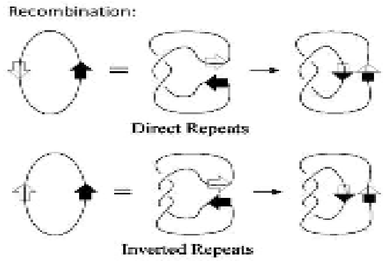
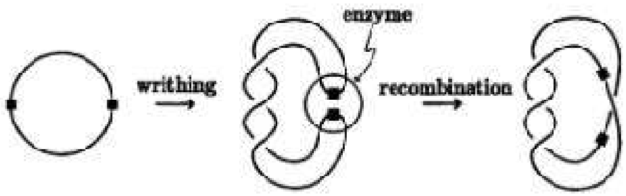


Figure 6.5



Substrate Synaptic complex Product

Figure 6.6 (site-specific recombination by enzyme)

It is seen that almost all the products obtained by the site specific recombination of trivial knot substrates are rational knots (or links) i.e. 2-bridged knots. Also the part of the synaptic complex acted on by an enzyme mathematically within a 3-ball is a (2,2) tangle.

The Tangle model of DNA

In the 1980s Claus Ernst and De Witt Sumners introduced a mathematical tangle model for the DNA complexes acted on by an enzyme.

The enzyme is considered as a 3-ball and the DNA acted on by the enzyme as properly embedded curves in the 3-ball. The DNA bound by the enzyme can then be divided into two parts, namely, the portion which is unchanged by the enzymes called ***substrate tangle*** denoted by S and that acted on by the enzyme called the ***site tangle***, T. The enzyme replaces the site tangle T with a new tangle called ***recombination tangle*** R. The numerator of the sum S+T is the substrate and that of the sum S+R is the product. This can be written as 2 equations in 3 variables,

N (S+T) = substrate

N(S+R) = Product

assuming that the substrate and product are known.

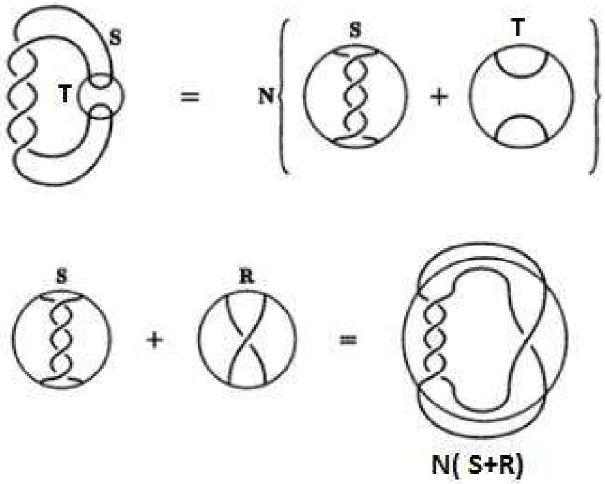


Figure 6.7

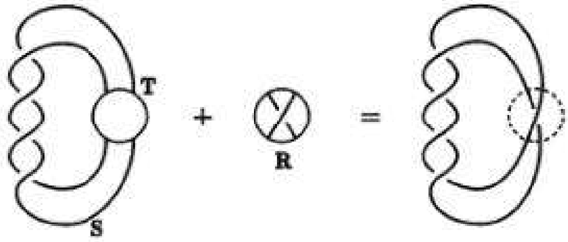


Figure 6.8 (enzyme action- site-specific)

It is seen that the enzyme Tn 3 resolvase, a topo-isomarase acts on a duplex cyclic DNA with direct repeats. It matches up the sites, replaces the T tangle with a single R tangle and then releases the molecule. Bio chemists determine the resulting product after enzyme action using a series of experiments. This was represented by the following equations using the notation for rational knots.

N(S+T) = N (1)	(Unknot)
N(S+R) = N (2)	(The Hopf Link)
N(S+R+R) = N (2 1 1)	(The figure 8 knot)
N(S+R+R+R) = N (1 1 1 1 1)	(The White Head Link)

From this set of equations, DeWitt Sumners and Claus Ernst proved that

S = (-3, 0) and R = (1).

They also proved that

N(S+R+R+R+R) = N (1 2 1 1 1) (The6_2 Knot)

This was shown to be true experimentally.

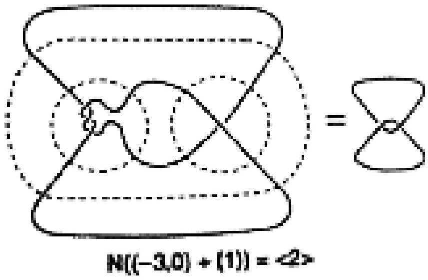


Figure 6.9

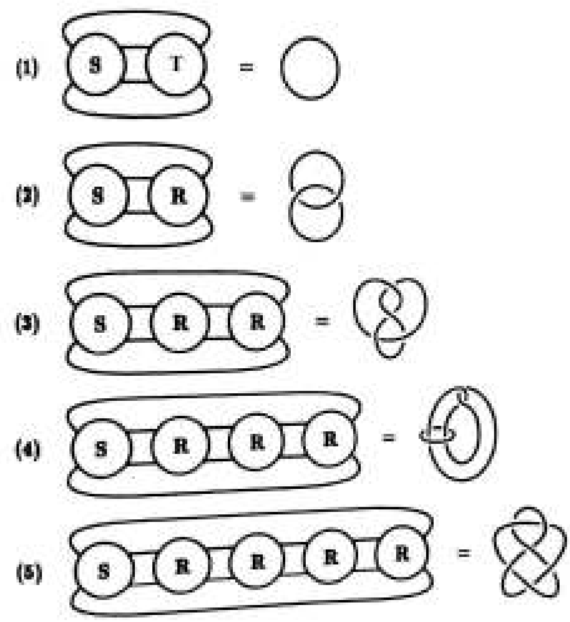


Figure 6.10

Modelling DNA topologically using the theory of knots can thus help to throw light into the technique of DNA recombination.

§. 6.2. Synthesis of knotted molecules

A molecule of DNA contains many atoms and is very complicated. There are some simple molecules that can be knotted or linked. A chain of same atoms, bonded in the same way can form a knotted chain or an unknot. Thus two or more molecules can be formed. These molecules may be distinct with different properties. For example, consider the two molecules in the figure 6.11.

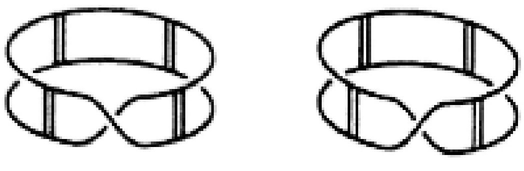


Figure 6.11 (Two molecules made of same set of atoms and bonds)

The first one is a Mobius band ladder with four rungs and right hand twist while the second one is with left hand twist. The two molecules are made up of the same atoms and bonds but have different molecular graph. The first embedding of the graph cannot be deformed to the second embedding of the graph through three dimensional space, that is, the two molecules are homeomorphic but are not isotopic.

A pair of molecules that are homeomorphic but not isotopic are called a pair of **topological stereoisomers**.

In the figure 6.12, same atoms are bonded in the same sequence to form three molecules. The first is an unknot, the second a left hand trefoil and the third a right hand trefoil. The three molecules are topological stereoisomers with each other.

Figure 6.12 (three topological stereoisomers)

Note: A left hand trefoil knot in distinct from the right hand trefoil knot.

Chemists were interested in topological stereoisomers because they could synthesize new molecules.

A set of linked molecular rings is called a **catenane**. In 1960, chemists first successfully synthesized a catenane. After the creation of a nontrivial link, chemists synthesized a knotted molecule in 1988. Wasserman and van Gulick independently had suggested that synthesizing a Mobius ladder with extra twists and then breaking the rungs of the ladder, a knot or a link can be formed as in the figure 6.14.

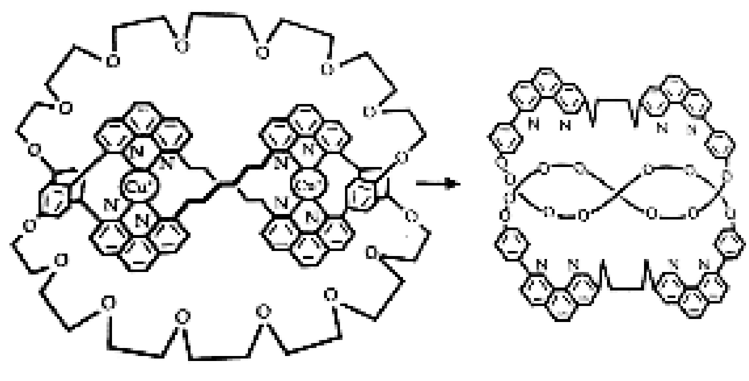


Figure 6.13 (The first synthesis of a knotted molecule)

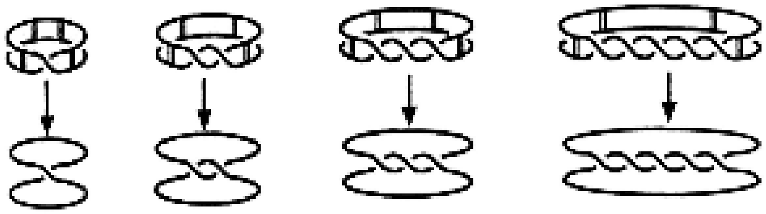


Figure 6.14 (Obtaining knots from twisted Mobius ladders)

But the molecule used to make the Mobius ladder was too rigid to allow the required twist. In 1990, Qun Yi Zheng (working under David Walba of the University of Colorado) added a clasp to the Mobius ladder which enabled him to synthesize a knotted molecule (Figure 6.15).

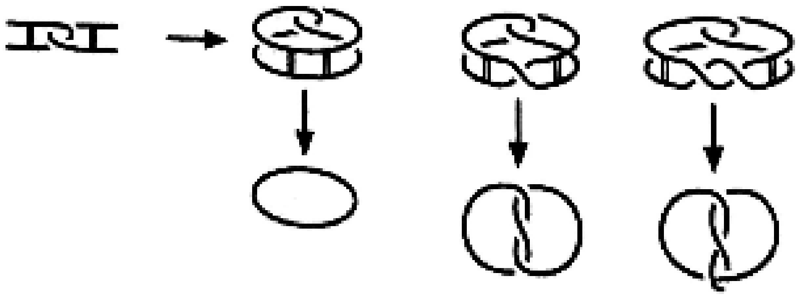


Figure 6.15 (Zheng's knotted molecule)

Van Gulick suggested three-strand ladders. Twisting, gluing together the two ends of the ladders and then breaking the rungs, it may be possible to synthesize knotted or linked molecules (Figure 6.16).

Figure 6.16 (Twisted three-strand ladders)

§. 6.3. Chirality of Molecules

As seen in the previous section, the left hand and the right hand Mobius bands with four rungs are topological stereoisomers (figure 6.11). One cannot be deformed to the other through space. Also, one is the mirror image of the other. A molecular graph which cannot be deformed through space to its mirror image is called **topologically chiral**. If it can be deformed through space to its mirror image then it is **topologically achiral.** (In knots or links it was called amphichiral).

A molecule may be topologically achiral, but not chemically achiral. The bonds between atoms may be too rigid to deform an actual molecule to its mirror image. However, a topologically chiral molecule must be chemically chiral.

In 1986, Jonathan Simon, a mathematician at the University of Iowa, proved that a Mobius ladder with four or more rungs is always topologically chiral. Thus any molecule with a molecular graph of a Mobius ladder with four or more rungs must have a topological stereoisomer. But this is not the case with a Mobius ladder with three rungs. John Simon demonstrated this using a figure (Figure 6.18).

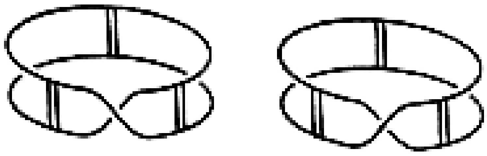


Figure 6.17 (Mobius band ladders with three rungs)

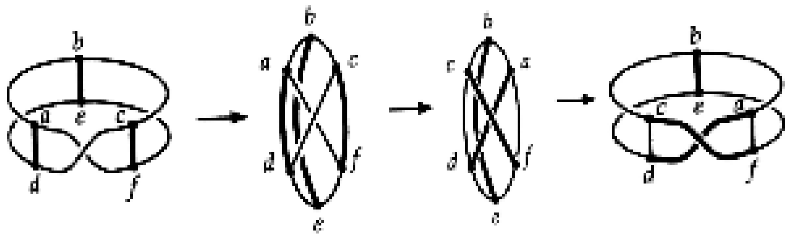


Figure 6.18 (John Simon's figure of a three-rung Mobius ladder)

Chemists are interested in knowing whether a knot is chiral or achiral in order to synthesize pairs of topological stereoisomers. A left handed chiral molecule may have properties different from that of the right handed one. This difference makes the study of chirality important in chemistry. For example, in a pharmaceutical company a drug made of a certain molecule may have an effect on a certain disease whereas that made from the mirror image of the molecule may have a side effect. Thus a method to distinguish chirality becomes essential during molecular synthesis.

Note: An alternating knot with odd number of crossings must be topological chiral.

CONCLUSION

I have made a survey of what has led to the study of knots in mathematics, the relevance of the study, the fundamental concepts which are dealt with in the theory, the slow development of the subject and its application in the inter disciplinary field. This is a brief report of a small area of the vast field of Knot Theory.

The knots can be studied classifying them as torus knots, satellite knots and hyperbolic knots. Every knot or link is proved to be a closed braid. Knots are being approached using the theory of braids as well. The concept of fundamental groups has helped in defining the Knot group. Thus Knot Theory can be studied connecting it with abstract algebra. A signed graph can be drawn corresponding to a link and vice versa. This provides a bridge between knot theory and graph theory.

The study of Reidemeister moves, some classical invariants like crossing number, knotting number, bridge number and other invariants like the genus of a knot and some polynomial invariants have been discussed here. The survey has shown that the fundamental problem of knot theory was the process of distinguishing knots. Many invariants have been discovered to show that two knots are not equivalent. If an invariant of two knots is equal it did not necessarily imply that the knots are equivalent. This necessitated further research on invariants. Also classifying knots and studying them with a topological point of view is becoming essential in the inter disciplinary field as well leading to a great scope to explore the field.

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